Time Inconsistency in Sovereign Debt Model

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Abstract

I provide quantitative analysis of the lack of credibility of Italian fiscal policy, by augmenting the standard model of sovereign default within incomplete markets to incorporate quasi-geometric time preferences and distortionary taxation. Quasi-geometric discounting is a simple and stylized way to capture both the political incentives of the government to deviate from an ex-ante optimal debt path and the appetite for immediate gratification. The core of the research is the identification of the preference parameters, which is achieved by exploiting the differences between time-consistent and time-inconsistent sophisticated governments when shaping fiscal guidance and interacting with pricing markets. While standard impatience leads to high default probabilities and low average debt accumulation, the quasi-geometric government has an incentive to "tie the hands" of future administrations, and it is willing to pay a price -in terms of suboptimal defaulting- to achieve that. Structural estimates for the discount factors are provided by matching relevant moments on Italian debt, spread, and fiscal policy. Results confirm the existing findings from lab experiments and show that time inconsistency is indeed a plausible explanation for the lack of credibility of Italian institutions.

1 Introduction

Choices under discretion and commitment have been widely studied in the monetary policy literature, with a specific emphasis on the consequences in terms of credibility and forward guidance. However, standard models of sovereign debt accumulation with optimal defaults as in [Arellano, 2008] and [Aguiar and Gopinath, 2006] do not include a description of the trade-offs that electoral promises and announcements generate. Nevertheless, the credibility of fiscal institutions proved often in the recent history of both developed and emerging countries to be fundamental in understanding and predicting the effects of debt accumulation and, more generally, of fiscal guidance. When interacted with incomplete financial markets, credibility becomes even more important in order to explain many of the differences between developed and emerging countries, by means of the forward-looking nature of sovereign spreads.

The lack of credibility under discretion may arise from a variety of political incentives as well as from preference aggregation mechanisms. For instance, when households inconsistently perceive future choices -i.e. change idea over time- and vote for candidates, it is reasonable to assume that their preferences will influence policy offers and the winner's strategy. This intuition has been investigated by [Bisin et al., 2015], that showed how hyperbolic discounting[1] may lead to high sovereign debt accumulation as the incumbent has an incentive to appease citizens' appetite for immediate gratification in a present-biased fashion: mechanisms such as Downsian competition, benevolence of the planner or a citizen-candidate are well suited to describe such an influence from private agents to public institutions.
Complementary, [Gollier and Zeckhauser, 2005] proved that hyperbolic discounting can be seen as the aggregation of heterogeneous time preferences and a representative agent displaying time inconsistency can be the result of a group of heterogeneous time-consistent agents with time-varying absolute risk aversion. This view does not require any "psychological" taste for a present bias and highlights how simple political pressures and lobbying mechanisms can be a sufficiently powerful to generate the lack of credibility of political institutions. Also, [Alesina et al., 1993] showed that inconsistency to announcements may well be generated by standard electoral cycles and asymmetric information: as elections approach, the incumbent highly values remaining in office and therefore is pressured to deviate from the ex-ante optimal path of debt consolidation.

Incentives to increase public expenditure above the optimal value, when anticipated by sophisticated administrations, create the scope for constraints on the action of the fiscal authority, that would otherwise be suboptimal if policies were credibly enforceable. To bring an example, [Alfaro and Kanczuk, 2017] show that fiscal rules would lead to a higher welfare, thanks to the restrictions that impose on deviations from the optimal path. Also, [Alesina and Tabellini, 1988] discuss how the intrinsic disagreement between current and future governments generates instability in the social choice function used to decide over the time profile of public spending: the result implies that the government has an additional reason to "tie the hands" of future governments.

Based on the intuitions above, this work addresses the lack of credibility of the fiscal authority by augmenting the standard model of sovereign debt accumulation within incomplete markets à la [Arellano, 2008] to incorporate quasi-geometric time preferences and distortionary taxation. Quasi-geometric discounting, which is the discrete-time counterpart of the hyperbolic discounting à la [Laibson, 1997], is a simple and stylized way to capture the political incentives of the government to deviate from an ex-ante optimal debt path, in a setting under discretion. Thanks to the assumption of sophistication, the government does not deviate in equilibrium, though it takes suboptimal decisions to counteract those incentives. That is, in order to prevent future administrations to deviate from the announced debt path, the current government delivers a large stock of debt to the next period, stifling additional borrowing by means of associated large increments in interests. However, such a "hand-tying" through debt bequeaths is costly; the price to be paid is in terms of defaulting less frequently than at the optimum. Future administrations would indeed face the same trade-off, between -on the one hand- additional borrowing or optimal defaulting and -on the other hand- imposing rigour to future selves. The difference with respect to the time-consistent (impatient) case is clear-cut, because for the latter the incentive to "tie the hands" does not play any role and impatience

[1] Hyperbolic discounting captures the intuition of a bias toward present that generates time inconsistency, i.e. an incentive to deviate from ex-ante optimal decisions. See the following sections for a more formal and complete discussion.

fully identifies tuples of default probabilities and average debt accumulations at any given business-cycle condition. In addition, hyperbolic discounting has the advance to capture impatience as well (i.e. high discount factors), which is generally known to be an essential feature in this class of models.

This research also provides an empirical counterpart to theoretical works such as [Jackson and Yariv, 2015], that study the aggregation of individual decisions by time-discounting individuals. The authors prove the necessity of time inconsistency in collective dynamic choices. The core of the research is indeed the identification of the two crucial preference parameters: the standard discount factor and the time-inconsistent component. As it is well known, estimating discount factors is a complex goal. In this case it is especially difficult as the effects of δ (standard factor) and β (time-inconsistent component) are substitutes. Nevertheless, the model is identified because the parameters are not perfect substitutes, thanks to the above differences between time-consistent and time-inconsistent sophisticated governments. Being the model fundamentally non-linear, standard techniques of estimation cannot be applied. However, point estimates for the preference parameters are provided applying the method of simulated moments as in [McFadden, 1989]. Results supports findings both from lab experiments and other structural estimates. For instance, the forthcoming paper by [Maxted et al., 2017] provides structural estimates for the two discount parameters using households' data on consumption and saving: results are even more conservative than those obtained here. In particular, using the benchmark (quarterly) calibration, the exponential and hyperbolic discounts equal respectively 0.98 and 0.66, when matching the moments on debt/GDP ratio, mean and volatility of spread, correlation between taxation and public expenditure with the cycle. The highly plausible point estimate of 0.98 for the "standard" discount factor is impressive, because even much more flexible models struggle to generate debt/GDP ratios sufficiently high without restricting discounting to be significantly lower than usual values.

This work contributes also to the literature that extended sovereign debt models for emerging countries to developed countries. Most of the recent attempts involved significantly extending the baseline version to include additional state variables, shocks and features, such as with long-term debt as in [Chatterjee and Eyigungor, 2012] and [Bianchi et al., 2016], fiscal rules as in [Anzoategui, 2016], news shocks as in [Durdu et al., 2013], productivity regimes and learning as in [Paluszynski et al., 2016]. Hyperbolic discounting, on the other hand, involves a simple one-parameter extension and provides the model with enough flexibility to generate realistic values for most of the variables of interest, while also accounting for differences between emerging and developed countries. The model is calibrated to Italian data during the sovereign debt crises (2011-2017) and performs well in explaining both the levels and the correlations of the relevant variables with the cycle.

It is therefore clear that the scope of the extension is far wider that it may seem at first sight: generalising time discounting opens to a parsimonious but flexible way to capture relevant aspects in both macroeconomics and political economy.

The work is divided in the following sections: section 2 discusses the existing literature on sovereign debt models and hyperbolic discounting. Section 3 describes in details the model. Section 4 explains the numerical solution used. Section 6 explains the benchmark calibration. Section 7 discusses the identification issues and presents the plots obtained simulating the model for different values of the time parameters. Section 8 presents the data used to match empirical and simulated moments, clarifies the strategy and show the results obtained. Section 9 concludes.

2 Literature review

The literature of time-inconsistent agents for behavioral economics starts with the seminal works of [Strotz, 1955], and is successively developed with an easily applicable setting through the works of [Phelps and Pollak, 1968], [O'Donoghue and Rabin, 2002], [Laibson, 1997] and [Gul and Pesendorfer, 2001], which led to interesting applications in the consumption models developed by [Angeletos et al., 2001], [Krusell et al., 2002] and [Krusell et al., 2010].

The first steps were taken by [Strotz, 1955] and [Phelps and Pollak, 1968] and their work gave new insights to the analysis of multiple equilibria and time-inconsistency of policies that was (and still is) a leading topic in many fields such as fiscal and monetary policy.
[Strotz, 1955] first suggested that people are more impatient when they make short-rum trade-offs than when they make long-run ones. Most of the empirical evidence that has directly experimented this theory has added some support to it.
The trade-offs so generated can be approached via Game Theory with the concept of Subgame Perfect Equilibrium (SPE) equilibrium between many selves. The first ones performing a similar solution were [Phelps and Pollak, 1968], who demonstrated that treating the problem as an intertemporal game between generations could lead to a solution which was easy to implement. The trade-offs were faced by different agents acting sequentially over different generations, one committing the actions of the following with their decisions.

Later the idea was extended by [Laibson, 1997], which focused the attention on a particular type of hyperbolic discounting (quasi-geometric discounting) with discrete time, easily tractable in some specific (but central) applications. His work, developed over the last decades, is still producing interesting results with empirical evidence in support of the hypothesis of the temptation of the agents.

In [Harris and Laibson, 2001] the authors showed in the context of a consumption-saving model that hyperbolic discounting implies non-standard properties for the value function, discontinuities in the policies, multiplicity of equilibria and absence of the global contraction property. Nevertheless locally most of the useful results are applicable as a consequence of a local contraction mapping theorem for hyperbolic preferences in the consumer's dynamic programming problem with uncertainty.

[Angeletos et al., 2001] developed a two-assets environment for consumption decisions and showed that an appropriate calibration with hyperbolic discounting is much more able to capture the propensity to consume out of increases in income (also temporary) and the allocation between liquid and illiquid assets over the life-cycle profile with respect to a standard model of perfect rationality and consistency.

A recent working paper ([Maxted et al., 2017]), based on [Laibson et al., 2000] estimated the value of preferences parameters using the method of simulated moments, necessary to capture the propensity to consume and the abnormal use of credit cards in the case of self-constrained agents. They showed that introducing some non-standard discounting helps the pattern of liquid and illiquid assets to be closer to the empirically observed quantities.

[Krusell et al., 2002] extended the theoretical analysis in [Harris and Laibson, 2001] showing a multiplicity of equilibria coexisting in the case of hyperbolic preferences à la Laibson and proposed a solution within the deterministic framework of a capital accumulation model. The authors' results indicate that the presence of a social planner without commitment may be even more harmful than its absence. This hass originated by the fact that myopic agents take the returns to savings as given, while the planner accounts for the effects of increased aggregate savings on the capital accumulation, leading to heavy undersavings.

[Krusell et al., 2010] studied the actions of a Pareto planner in a similar setting using the axiomatic time-consistent preferences introduced by [Gul and Pesendorfer, 2001] and investigated the equivalence with the model by [Angeletos et al., 2001]. Moreover, they showed that a positive subsidy to investments (i.e. an active fiscal policy) would lead to higher aggregate welfare in a context with a benevolent planner.

[Judd, 2004] showed that the local result offered by [Harris and Laibson, 2001] can be extended to other frameworks and in general to every "nearby" game[2]. The author proves the theorem using the generalized Euler equation, but his approach can be extended to cases where the Euler equation does not exist, through the value function. Moreover Judd provides an example that includes a comparison between different numerical algorithms.

[2] I.e. every game that is a perturbation of a game with solution obtainable from time iteration.

[O'Donoghue and Rabin, 2002] introduced the naive case along with the sophisticated one, for agents who are constantly fooled by their wrong perceptions about their deviations from the exponential preferences. Sophisticated agents on the contrary behave knowing exactly that they will be tempted to deviate in the future. In the presence of some forms of commitment they will then be willing to pay to reduce their possibility set only to the "ex-post" optimal choices. Naive agents otherwise do not perceive entirely this necessity and their commitments reach much lower values; however then they will succumb to temptation and regret their decisions.

Interesting empirical evidence of hyperbolic discounting is offered by a number of different research studies, including lab experiments. Laboratory experiments often ask subjects to weigh immediate rewards against delayed rewards. Results often show inconsistencies between the choices, implying that agents display decreasing discount rates. Field data also supported the thesis, as shown by works such as [Vigna and Malmendier, 2006] and [Chesson and Viscusi, 2000].

Despite the recent growing attention to bounded rationality, one of the few articles that theoretically investigates the influence of bounded rationality in government's decisions is [Bisin et al., 2015].
The authors present a model of accumulation of public debt under different assumptions of voting mechanisms. If the voters have self-control problems, implying that they would try to commit using illiquid assets, in equilibrium the government accumulates debt to appease the desire for immediate gratification and being re-elected. Voters act strategically anticipating this deviation from their self-constricted optimal path and rebalance their portfolio allocating even more wealth in illiquid assets. This in turns creates a loop in which the government increases the debt more and more. The lower is the debt limit, the higher the welfare that agents can achieve. Absent a debt limit, debt accumulates without bounds in infinite periods. The debt ceiling is then claimed to be necessary to decrease inefficient distortions which the economy must incur to repay the large accumulated debts at equilibrium. When debt limits are too high, agents' ability to commit is impaired.

The literature on sovereign debt models builds from the seminal works of [Eaton and Gersovitz, 1981], [Aguiar and Gopinath, 2006], and [Arellano, 2008]. Recently, the topic has received increasing attention and a large body of research has been produced. For a complete and up-to-date survey see [Aguiar et al., 2016]. The closest researches to this work are probably [Aguiar and Amador, 2011] and [Alfaro and Kanczuk, 2017]. The latter provides a quantitative analysis of fiscal rules in a standard model of sovereign default with hyperbolic discounting, using Brazilian data. [Aguiar and Amador, 2011] extend [Aguiar and Gopinath, 2006] and propose a variant of the open economy neoclassical growth model that emphasizes political economy and contracting frictions. The political economy frictions are described through quasi-geometric discounting, while the contracting frictions act as a lack of commitment regarding foreign debt and expropriation. In particular the authors investigate the relationship between political environment and growth, debt accumulation and the evolution of capital taxation. Nevertheless, up to my knowledge, this is the only work that empirically estimates and identifies time inconsistency as a form of lack of credibility in a standard sovereign default model.

3 The Benchmark model

The benchmark model builds from [Arellano, 2008] and [Cuadra et al., 2010], and features an economy with domestic households, a domestic government and foreign lenders.
Households enjoy private and public consumption and dislike working. In addition they feature the desire of immediate gratification in the form of quasi-geometric discounting when considering trade-offs between present utility and future streams of utilities. No private savings or capital investment are directly chosen by households, but aggregate savings are present in the model through the trade balance and debt accumulation/repayment. This implies that consumers face a static problem that can be maximized out in the intertemporal governmental problem. The government is benevolent but shares the same myopic perspective as the consumers. Government can decide the fiscal policy, composed of a distorsive tax on consumption and public spending that directly enters the utility of the representative household. Moreover the government

can decide how much debt to accumulate in order to pay for the public expenditure. Each unit of debt is borrowed from abroad according to a price that is decided in equilibrium and is the result of the default risk, function of the state of the economy and the amount of debt. The asset market is incomplete as the only asset available is a government bond that takes the form of a one-period discounted bond. This bond indistinctly pays one unit of the public consumption good at the beginning of each following period, and is bought to its equilibrium price in each period. Debt can increase or decrease in amount from period to period according to how much new debt is issued, as all the debt is repaid in every period. The government can also decide not to repay the outstanding debt, hence defaults play the role of an imperfect -and very inefficient- insurance against downturns. In the case of default the economy suffers a productivity cost and is excluded from international markets for a random number of periods. Foreign lenders are risk-neutral and borrow in a perfectly competitive market charging a risk premium that accounts for the default risk they face.

Default risk is higher in recessions as repaying of debt is more difficult when productivity is low, implying a negative correlation between gross domestic product and sovereign debt spread. Therefore in bad times the trade-off between providing public consumption through debt and taxation becomes harsher and the government will sometimes decide to default and rely on fiscal policy alone. This creates an incentive for a procyclical taxation and spending. At the same time during a recession, private consumption is lower and utility smoothing over the life-cycle implies that there is an incentive to raise the debt, even at higher costs, in order to bring total consumption back to the average level. This creates an incentive for a countercyclical fiscal policy. The balance between the two effects generates interesting trade-offs that will occasionally result in procyclicality or countercyclicality according to the values of the calibrated parameters. Introducing hyperbolic discounting brings more arguments in favour of the countercyclical policy, as additional costs of debt are more bearable when immediate gratification is taken into consideration. This in turn would imply that more often the policy chosen by the government will be countercyclical, with a negative correlation of the total expenditure (public consumption minus taxation) to gross domestic product. However, at the equilibrium, government will suffer from tight budget constraints, and the room for countercyclical actions may completely disappear.

3.1 Households

The model has a representative household with preferences over the stream of future consumption and public spending bundles:

$$\mathbf{E_0}\left\{U(C_0, G_0, l_0) + \beta \sum_{t=1}^{\infty} \delta^t U(C_t, G_t, l_t)\right\} \tag{1}$$

The utility U is a CRRA, aggregated between consumption and public spending separably:

$$U(C, G, l) = \pi \left[\frac{\left(C - \frac{l^{1+\omega}}{1+\omega}\right)^{1-\sigma}}{1-\sigma}\right] + (1-\pi)\left[\frac{G^{1-\sigma}}{1-\sigma}\right] \tag{2}$$

where C and G are private and public consumption, while l is endogenous labour. Instantaneous utility is concave, strictly increasing and continuously differentiable.
$\beta \in [0, 1]$ is the quasi-geometric discount factor that captures the preference for immediate gratification, as discounting only future streams of utility. $\delta \in [0, 1]$ is the standard discount factor that is exponentially compounded.
The household maximization problem is constrained by the budget set over consumption and leisure choices:

$$(1 + T_t)C_t = \Phi_t + w_t l_t \tag{3}$$

where Φ_t are the profits of the firms, given by:

$$\Phi_t = Z_t F(l_t) - w_t l_t \tag{4}$$

and w_t are the salaries that result from the profit maximization of the firms together with the optimal labour choice $w_t = Z_t F'(l_t)$.
The problem can then be expressed as:

$$\max_{\{C_t, l_t\}_{t=0}^{\infty}} U(C_0, G_0, l_0) + \mathbf{E_0} \left\{ \beta \sum_{t=1}^{\infty} \delta^t U(C_t, G_t, l_t) \right\} \tag{5}$$

$$s.t.\ (1 + T_t)C_t = Z_t F(l_t) \tag{6}$$

$$C_t \geq 0 \tag{7}$$

$$l_t \geq 0 \tag{8}$$

Optimal choices of labour and consumption of the household can be maximized out in the government's problem and are obtained from the equivalence between the ratios of marginal utilities of consumption and leisure and the prices of consumption (taxation) and leisure (wage).
Let $Z_t F(l_t) := Z_t l_t$, then optimal labour is given by the equation:

$$l_t^{\star} = \left[\frac{Z_t}{1 + T_t} \right]^{\frac{1}{\omega}} \tag{9}$$

From the budget constraint, optimal consumption is residually obtained as:

$$C_t^{\star} = \left[\frac{Z_t}{1 + T_t} \right]^{1 + \frac{1}{\omega}} \tag{10}$$

3.2 Government

The government is composed of citizen-candidate politicians, which maximize the utility of the households. The decisions that the government can take are about fiscal policy (T_t and G_t) and debt (B_t).
The government can promote a one-period discounted non-contingent bond at the price q_t to obtain $q_t B_{t+1}$ units of public consumption in the next period. Therefore markets are incomplete.
In each period, after observing the realization of the productivity and the debt inherited from the previous year, the government can decide how much new debt to issue or to default and start afresh with no outstanding debt.
This choice depends on the comparison with the "reservation" utility that comes from defaulting and not having access to the market for a random number of periods.
If default is not optimal, the government can then choose the debt policy and the fiscal policy. Otherwise when default is optimal, it can decide only taxation and public consumption.
The comparison takes the above into account together with the cost of productivity associated with the default.
Let us define the current value function $W(Z_t, B_t)$ of the productivity shocks Z_t and amount of assets borrowed B_t as the maximum between the current value functions of staying in the market and defaulting:

$$W(Z_t, B_t) = \max \left\{ W^m(Z_t, B_t), W^d(Z_t) \right\} \tag{11}$$

where d and m indicate the choice of default or market and the argument Z_t of $W^d(Z_t)$ is a short notation for $h(Z_t)$, the value of productivity in autarky.
The default decision is defined by the rule:

$$D(Z_t, B_t) = \begin{cases} 1 & \text{if } W^m(Z_t, B_t) < W^d(Z_t) \\ 0 & \text{else} \end{cases} \tag{12}$$

The default rule defines a set of states $\{Z, B\}$ in which the government will always choose to default. This set of default states determines uniquely the price of the discounted bond changes. It is important to notice that the default rule maps only known quantities at time t into an indicator function that imposes default

or commitment. Once the state of the economy, which is composed of productivity value and level of debt, is observed, defaulting or not is not a choice but a compulsory rule. This is necessary to ensure rationality of the agents at the equilibrium.
Once the decision is taken to default or not, the government can choose the fiscal policy and, in case, the new debt level.

Let us first analyse the case in which the government does not default. New debt, taxation and expenditure can be decided according to the budget constraint:

$$G_t + q(Z_t, B_{t+1})B_{t+1} = T_t C_t + B_t \tag{13}$$

where the quantity $TB(B_{t+1}, B_t) = q(Z_t, B_{t+1})B_{t+1} - B_t$ is the trade balance.
When choosing optimal taxation, expenditure and debt the government takes into account the optimal choices of the consumer. Therefore the problem in the case of access to market can be rewritten as:

$$W^m(Z_t, B_t) = \max_{\{T_t, G_t, B_{t+1}\}_{t=0}^{\infty}} \left[U(C_t^\star, G_t, l_t^\star) + \beta\delta \int_{\mathcal{Z}} V(Z_{t+1}, B_{t+1}) d\pi(Z_{t+1}|Z_t) \right] \tag{14}$$

$$s.t.\ l_t^\star = \left[\frac{Z_t}{1+T_t} \right]^{\frac{1}{\omega}} \tag{15}$$

$$C_t^\star = \left[\frac{Z_t}{1+T_t} \right]^{1+\frac{1}{\omega}} \tag{16}$$

$$G_t = T_t C_t^\star + B_t - q(Z_t, B_{t+1})B_{t+1} \tag{17}$$

$$G_t \geq 0 \tag{18}$$

$$T_t \geq 0 \tag{19}$$

$$B_{t+1} \leq 0 \tag{20}$$

The variables with the star represent the optimal choices of the consumer, that have been taken as given in the maximization problem of the planner. The probability measure used to calculate the integral over the set of values of Z, $\mathcal{Z}$, is derived from the law of motion of productivity, which will be discussed later on. V is the continuation value function from the perspective of the current self t corresponding to the associated "nearby" problem with $\beta = 1$ from $t+1$ onward.

Let us now consider the case in which the government defaults.
The government is temporarily excluded from the market; the length of this period depends on a parameter λ which represents the probability of re-entry. The productivity Z_t suffers a negative shock that is captured by the function $h(Z_t)$.
The problem in autarky becomes then:

$$W^d(Z_t) = \max_{\{T_t^d, G_t^d\}_{t_0}^{\infty}} \left[U(C_t^{d\star}, G_t^d, l_t^{d\star}) + \beta\delta \int_{\mathcal{Z}} \lambda V(Z_{t+1}, 0) + (1-\lambda)V^d(Z_{t+1})\ d\pi(Z_{t+1}|Z_t) \right] \tag{21}$$

$$s.t.\ l_t^{d\star} = \left[\frac{h(Z_t)}{1+T_t^d} \right]^{\frac{1}{\omega}} \tag{22}$$

$$C_t^{d\star} = \left[\frac{h(Z_t)}{1+T_t^d} \right]^{1+\frac{1}{\omega}} \tag{23}$$

$$G_t^d = T_t^d C_t^{d\star} \tag{24}$$

$$G_t^d \geq 0 \tag{25}$$

$$T_t^d \geq 0 \tag{26}$$

Here private consumption and labour are optimal given the reduced value $h(Z_t)$ of productivity. Taxes are the only means to repay the public consumption and budget has to break even in every period. Choices of

policies weigh the possibility of re-entering in the market with the possibility of staying in autarky also in the following period.
Again V and V^d are the continuation value functions, corresponding to the value functions of the associated "nearby" problem with $\beta = 1$ from $t + 1$ onward.

3.3 Foreign Lenders

Foreign lenders act in a perfectly competitive market of bond lending. Since the setting is a small open economy, prices cannot be influenced strategically and will be the result of the equilibrium.
Foreign debt is expressed in terms of the consumption good. I assume that the law of one price holds for the consumption goods and the nominal exchange rate defined as the domestic-currency price of one unit of foreign currency is normalized to be constant at one.
Lenders enjoy a reservation utility by trading at the risk-free interest rate r_f.
By assumption they are risk-neutral and maximize their expected profits in every period. Due to competition of markets in the maximization the profits will be equalized to zero, equating gains and costs of lending. Profits are given by:

$$\Phi_t = -q(Z_t, B_{t+1})B_{t+1} + \frac{1 - \Pi(Z_t, B_{t+1})}{1 + r_f} B_{t+1} \tag{27}$$

$-q(Z_t, B_{t+1})B_{t+1}$ is the amount that is provided by the lenders to the government, which represents the explicit cost of the transaction. $\frac{1-\Pi(Z_t,B_{t+1})}{1+r_f} B_{t+1}$ are the revenues of the transaction, weighted for the endogenous probability of receiving the payment $\Pi(Z_t, B_{t+1})$. Note that this probability depends on the state of the economy and the choice on debt.
Perfect competition implies that when profits are maximized the price of the bond reflects the probability of default:

$$q(Z_t, B_{t+1}) = \frac{1 - \Pi(Z_t, B_{t+1})}{1 + r_f} \tag{28}$$

That probability is calculated averaging over all the future productivity values -i.e. expected states of the economy-, assigning unit mass when default occurs and zero mass when it does not. That is:

$$\Pi(Z_t, B_{t+1}) = \int_{\mathcal{Z}} D(Z_{t+1}, B_{t+1}) d\pi(Z_{t+1}|Z_t) \tag{29}$$

where $D(Z_{t+1}, B_{t+1})$ is defined as in (12).
Notice here that the rule is shifted one period forward: the integral represents how the government is expected today to behave tomorrow. Values of future productivity are weighted using the probability defined by its law of motion.
When Z_t (productivity today) is such that it is impossible that default will occur tomorrow, then the cost of the bond will be equal to $\frac{1}{1+r_f}$. Symmetrically when the level of debt chosen B_{t+1} is low enough the spread with the risk-free bond will be zero.
When looking for a steady-state solution of the model time indexes will be dropped[3]:

$$\Pi(Z, B') = \int_{\mathcal{Z}} D(Z', B') d\pi(Z'|Z) \tag{30}$$

and the bond price will not depend on time:

$$q(Z, B') = \frac{1 - \Pi(Z,B')}{1 + r_f} \tag{31}$$

[3] I will use the notation without time indexes to indicate the steady state solution obtained taking limits of the backward induction (also called finite-horizon equilibrium selection). Rigorous discussions regarding the existence of the steady state are left for future works, but simulations show the time convergence of the paths of variables (See the section about the identification).

3.4 Productivity

The law of motion of the productivity follows standard assumptions. It is defined as a logarithmic autoregressive process of first order

$$\ln(Z_t) = \alpha \ln(Z_{t-1}) + \varepsilon_t \tag{32}$$

where $\varepsilon_t \sim \mathbf{N}(0, \sigma_\varepsilon^2)$ and $\alpha \in [0, 1)$.
Assuming a this dynamic for the productivity fully specifies the probabilities used in the integral $\pi(Z_{t+1}|Z_t)$ using standard results on the normal distribution[4].

Productivity in autarky is decreased to capture the costs of defaulting, that is households produce fewer goods with the same amount of labour service. The loss is specified following [Arellano, 2008] in the form of a ceiling:

$$h(Z_t) = \begin{cases} \phi \mathbf{E}(Z_t) & \textit{if } Z_t > \phi \mathbf{E}(Z_t) \\ Z_t & \textit{else} \end{cases} \tag{33}$$

3.5 Equilibrium

[Krusell et al., 2002] showed indeed that, typically, there is a problem of indeterminacy of Markov-perfect equilibria in an infinite-horizon economy. [Auclert and Rognlie, 2016] argues that Markov perfect equilibrium is unique in the canonical infinite-horizon model with Markov income and permanent autarky punishment. Even if the author proved uniqueness in the case of temporary shock and i.i.d. income process, this model involves both temporary autarky punishment and Markov income, hence cannot fully rely on the above result. Therefore, following [Hatchondo et al., 2009], I will focus on limits of subgame perfect equilibria (SPE) in stationary Markov strategies. The recursive equilibrium for every subgame can be defined as the following:

Definition A recursive equilibrium for this economy is characterized by:

1. A set of continuation value functions $\{V, V^d\}$ and current value functions $\{W, W^m, W^d\}$,
2. A set of policy functions for household's consumption $\{C, C^d\}$ and household's labour supply $\{l, l^d\}$,
3. Policy functions for the government's default decision D, optimal asset holding B', optimal government expenditures $\{G, G^d\}$ and tax rates $\{T, T^d\}$,
4. A bond price function q,

such that

1. Given the government's policies and the bond price function, then household's policies solve the household's problem,
2. Given the bond price function and the optimal policies for the household, the set of current and continuation value functions and the policy functions of the government solve the problems in autarky and market,
3. The equilibrium bond price equals the expected profits of the international lenders to zero.

[4] see [Tauchen, 1986] for a complete explanation.

4 Calibration

Table (1) presents the values used in the benchmark calibration of the model. All the values chosen are standard or taken from the literature and are compatible to estimates for Italy.

The final consumption of goods of households accounts for 62% of the GDP and 76% of the public and private final consumption. Final consumption of goods of the government accounts for 20% of the GDP and 24% of the public and private final consumption. The German real interest rate for a 3-months discount bond in the period 2011Q1-2017Q1 equals -0.05% (approximately zero). In the calibration the 3-months risk free interest rate has been set to 0%. The AR(1) coefficient estimated using the seasonally adjusted Italian real GDP equals 0.87, in line with the calibration of the productivity shocks. The value for the volatility of the shocks (6%) is standard in the literature and similar to the choices in [Cuadra et al., 2010] and [Alfaro and Kanczuk, 2017]. That value significantly affects volatilities of variables (spread, GDP, consumption, fiscal policy), but is nevertheless about 3 times higher than estimates for the productivity shocks. However, a high volatility is necessary to the model to generate sufficient variations in the variables and to capture the cyclicality of the spreads and debt. In particular a high total discount rate tends to shrink the model's variability and collapse choices to corner solutions. Since this is the object of the research, inconsistency of those standard deviations with data is pardoned.
Also the default penalty is a sensitive parameter and severely affects the levels of the variables. The value chosen has the target to match realistic levels of the debt/GDP ratio, given the choice for the time parameters. Practically the values that can be chosen are limited and are included in the range [0.9,1), as out of this interval the government always or never defaults. Within the range the parameter does not significantly affect the level of the variables. In addition, the value is standard in the literature for similar models, such as [Schmitt-Grohé and Uribe, 2016] and [Alfaro and Kanczuk, 2017].

Calibration

Description	**Parameter**	**Value**	**Target**
Risk Aversion	σ	2	[Aguiar and Gopinath, 2006]
Inverse Frish elasticity	ω	0.9	Standard
C weight	π	0.7	[Cuadra et al., 2010]
Re-entry probablity	λ	0.1	[Gelos et al., 2011]
Z process AR(1)	α	0.85	[Cuadra et al., 2010]
Z process volatility	σ_ε^2	0.06	[Cuadra et al., 2010]
Risk-free interest	r_f	0	Estimates for 2011Q1-2017Q1
Default penalty	ϕ	0.9	[Alfaro and Kanczuk, 2017]

Table 1: Values of the parameters in the benchmark calibration.

As shown in [Alfaro and Kanczuk, 2017], the risk aversion is a central parameter affecting the degree of cyclicality of the variables in the fiscal policy. This is also confirmed by the work of [Maxted et al., 2017], that provided estimates for it. In addition, with some specific functional forms for the CRRA utility with $\sigma = 1$ it can be proven that the model with hyperbolic discounting collapses to the corresponding model with exponential discounting, as the role of utility smoothing becomes prevalent over the appetite for immediate gratification. A standard calibration with $\sigma = 2$ is then chosen to balance properly the two effects.

5 Identification

This sections presents the identification strategy for the preference parameters of the government. Summary statistics and policies are plotted against different values of the hyperbolic and exponential discount factors for given calibrations, in order to compare the effects of the two discounts on the variables of interest. Higher values of discounting are in general associated with an increased value attributed to present (private and public) consumption. This effect leads to a higher accumulation of debt to anticipate consumption and higher probabilities of default. Both lower values of δ and β would serve this purpose. However trade-offs between present and future are perceived differently in the two cases: a standard agent would be persuaded that its choice is always optimal, a behavioural agent would fight against future selves to impose its perspective. Rational market forces interact with those agents assigning gains and punishments in the form of asset prices and reward differently the two behaviours: due to this interaction it is possible to better understand the individual effects of the two forms of discounting. From the simulations it clearly arises that a patient but hyperbolic agent has a strong incentive to accumulate and stabilize debt, while a rational impatient agent (with equal short-term discount rate) would rather procyclically accumulate debt and default often on its obligations when negative shocks realize. The stabilization is a consequence of the desire to impose rigour on future solves, by mean of a costly hand-tying. The commitment can be restored thanks to the interactions with the lenders, that assign harsh punishments to deviations from the path of debt consolidation.

The intuition behind identification is threefold. First, short-term incentives are identical for both types. Higher discounting, either exponential or hyperbolic, makes it easier to default in downturns or accumulate additional debt to increase immediately consumption. If cumulated liabilities are too high and productivity shocks make it difficult to repay without a substantial drop in consumption or jump in taxes, default appears to be an easy way out. The more the government is impatient for immediate consumption, the more it will be tempted by default on obligations. In this case, both types of discounting move equilibrium levels of variables in the same direction: the higher the impatience, the higher the probability of default and the faster the debt accumulates in downturns. Second, long-term incentives are different, being the hyperbolic type more patient. This implies that costs that will be paid far in the future, such as productivity costs in autarky, are relatively much more effective for the patient but time-inconsistent type. Hence, the incentive to default greatly reduces and higher levels of debt are sustainable. Third, the inconsistent type has an incentive to tie the hands of future selves. This in turn reduces further the incentive to default at each level of debt, but also moves the desired level of debt right below the default threshold, cutting out future selves from additional borrowing.

This behaviour can be visualised in Figure (1): even if at each time horizon the short-term discounts are equal, the long-term discounts differ and, in particular, $\beta\delta$- discounter strikes to impose itself more conservatives actions.

The main point is that an impatient government is not afraid of defaulting, as it internalizes only a small part of the future costs that it will bear in order to enjoy immediate gains. Lenders perfectly foresee this careless behaviour and punish it with large spreads. At equilibrium, default happens very often also on low liabilities, as soon as small downturns realize. Hence, debt accumulation is volatile but the average debt level weights the sudden increases with the frequent periods of autarky (in which debt is zero), and therefore is low on average. On the contrary, the inconsistent government is highly concerned about persistent future costs, despite highly values current consumption. Hence, for any given level of debt, default probability is low and interest are moderate as well. Nevertheless, short-term impatience and relatively low spreads lead to debt accumulation, until interests increase to counteract the benefits of additional borrowing. In turn, high interests play the role of the desired commitment tool that ties the hand of future selves, as long as debt does not exceed the default threshold: this is the price to be paid, in terms of suboptimal defaulting, in order to foster patience of future administrations.

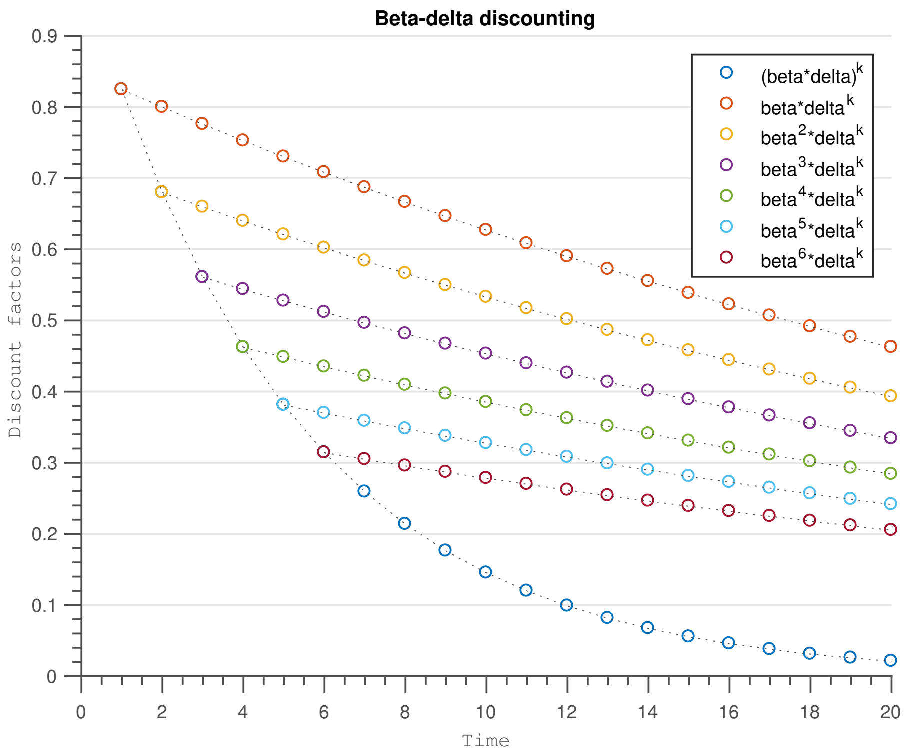

Figure 1: Plot of values of $\beta\delta^k$ and $(\beta\delta)^k$, for $\beta = 0.85$ and $\delta = 0.97$.

The difference between the two types can be visualized in Figure (2) and (3), which display the convergence to the time convergence of debt level and spreads to the equilibrium. The punishment mechanism provided by bond pricing controls for default probabilities to go wild. However in general with an impatient standard agent the threshold between the default trigger and repayment is much more unstable: market incompleteness ensure imperfectly against productivity shocks and force considerations in terms of expectations. The steady state of the hyperbolic government is on the contrary stable, due to the joint intervention of self-commitment and pricing markets.

Summary statistics

	β	1.0000	0.8500
	δ	0.8245	0.9700
Correlations	GDP - Trade balance	-0.526	0.276
	GDP - Spread	-0.384	-0.325
	GDP - Default Probability	-0.142	-0.097
	GDP - Government transfers	0.527	-0.263
Means	TB/GDP %	0.144	-0.082
	GDP	0.791	0.793
	Debt/GDP %	19.12	34.54
	Spread - Basis points	55.51	9.29
	Default Probability %	5.49	0.90
	Debt Growth/GDP %	-0.18	0.06

Table 2: Summary statistics for simulations with comparable discount factors.
Calibration of other parameters is the benchmark. The first 100 quarters are burned, 50 quarters are averaged for 10 000 agents.

This difference between time consistency and inconsistency translates into a higher default risk for the exponential discounter, higher debt costs and therefore a lower absolute value for debt. Simulations for comparable values of the discount factors are presented in Table (2). The results show the intuition above stated: even if the short-term discount factors are effectively equal, the market punishment creates different incentives for debt accumulation. The exponential discounter (left column) accumulates a little bit more than half the debt/GDP ratio accumulated by the hyperbolic discounter but has a probability of default of 5.5%, much higher than the 0.9%. Spreads reflect these two facts, and are more than 5 times higher.

5.1 Steady state

Figures (2) and (3) show the evolution of the means of debt to GDP ratio and debt growth, spread and default probability for the two simulations in Table (2). It is immediate to notice that the debt of the hyperbolic government has positive slope and converges concavely to an equilibrium level, whereas, for such a high rate, the exponential discounter is not stable at equilibrium (debt growth/GDP equals -0.4% on average) and the level decreases steadily. We will see that this is a consequence of the punishment mechanism performed by the bond pricing, which implies a high instability with either a low debt for realizations higher than the median of the distribution of the productivity shocks or a default on obligations for negative shocks. The path completely changes for lower values of the discount rate (Figure (4)) and the convergence to a steady state value is similar to the case in Figure (2).
In the simulations initial level for state variables are the median value of productivity, market access and no starting debt.

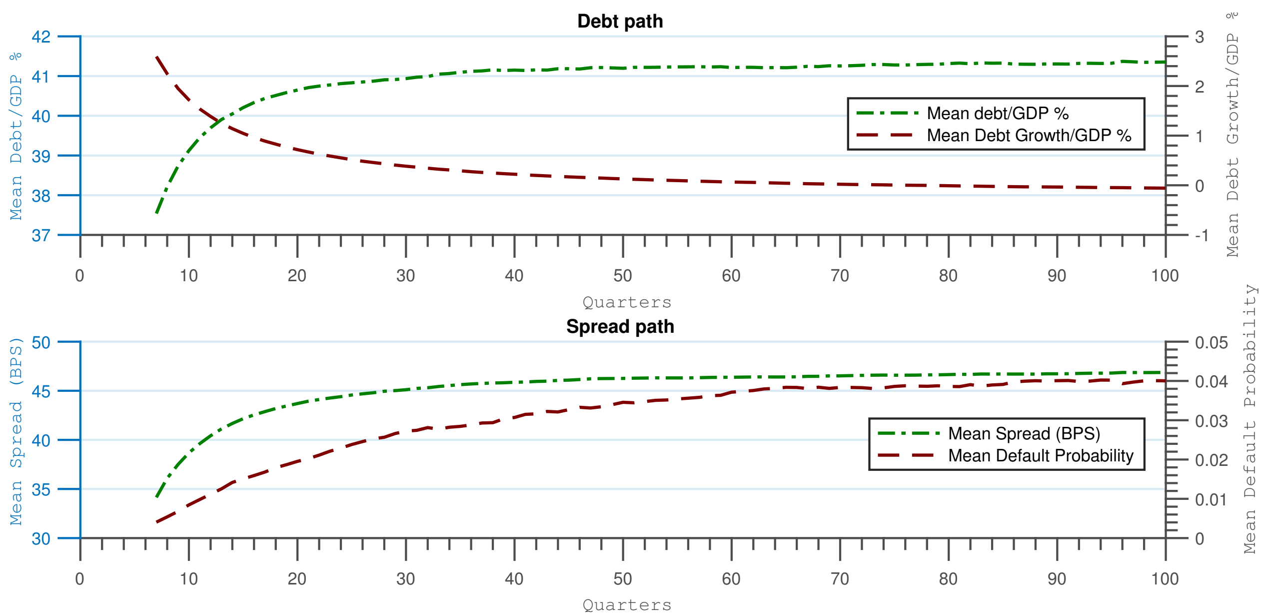

Figure 2: Plots of the paths of debt and spread for different simulation time lengths; $\beta = 0.66$ and $\delta = 0.98$, without burn period and 10 000 simulated agents. Initial values for productivity is the median, with market access and no starting debt. Paths are calculated simulating the variables for different numbers of quarters and then taking averages.

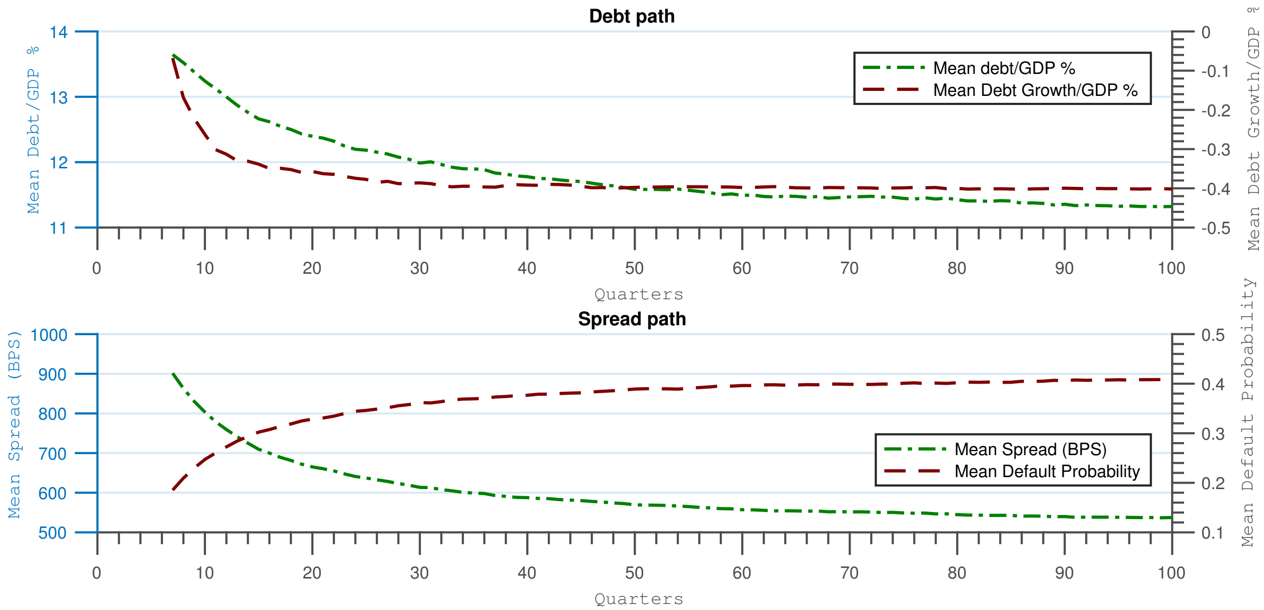

Figure 3: Plots of the paths of debt and spread for different simulation time lengths; $\beta = 1$ and $\delta = 0.6468$, without burn period and 10 000 simulated agents. Initial values for productivity is the median, with market access and no starting debt. Paths are calculated simulating the variables for different numbers of quarters and then taking averages.

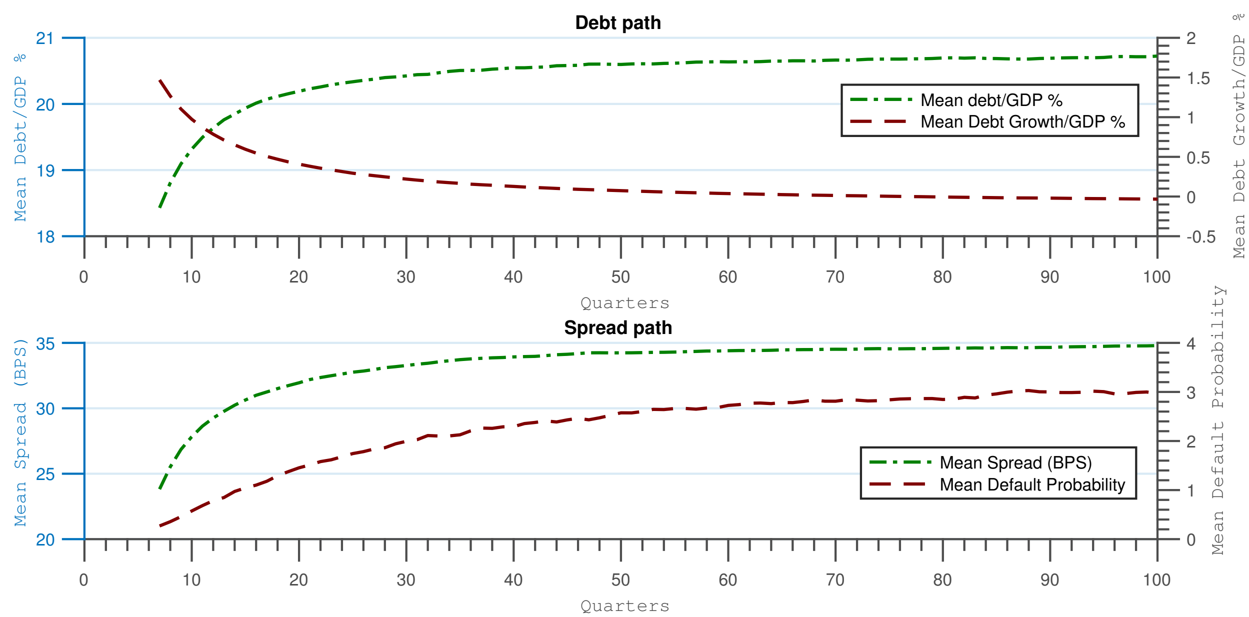

Figure 4: Plots of the paths of debt and spread for different simulation time lengths; $\beta = 1$ and $\delta = 0.85$, without burn period and 10 000 simulated agents. Initial values for productivity is the median, with market access and no starting debt. Paths are calculated simulating the variables for different numbers of quarters and then taking averages.

5.2 Debt and spread

Figures (5) and (6) display the summary statistics relative to debt and spread against different values of the discount factors on the x-axis. Figure (5) shows the result while changing the value for β, keeping the benchmark calibration and $\delta = 0.98$. Figure (6) has the same calibration but now $\beta = 1$ and delta changes. It is clear that for equal values of the discount factor between today and tomorrow (i.e. when $\beta\delta$ from the first panel equals δ in the second), the effect of δ in the latter is much stronger on the levels of the variables. For instance consider the data point corresponding to $\beta = 0.85$; in the second panel the corresponding discount is $\delta = 0.98 \times 0.85 = 0.833$. In Figure (5) spread level is about 8 BPS while in Figure (6) spread level is about 100 BPS. Similarly percent spread volatility (normalized dividing by the mean spread) is about 2% in the former and about 10 % in the latter. In general absolute spread volatility is not heavily affected by changes in β, while it is by changes in δ. The level of rescaled spread volatility -i.e. divided by the spread mean- is twice as high in the second figure than in the first, confirming the intuitions above also in relative terms.

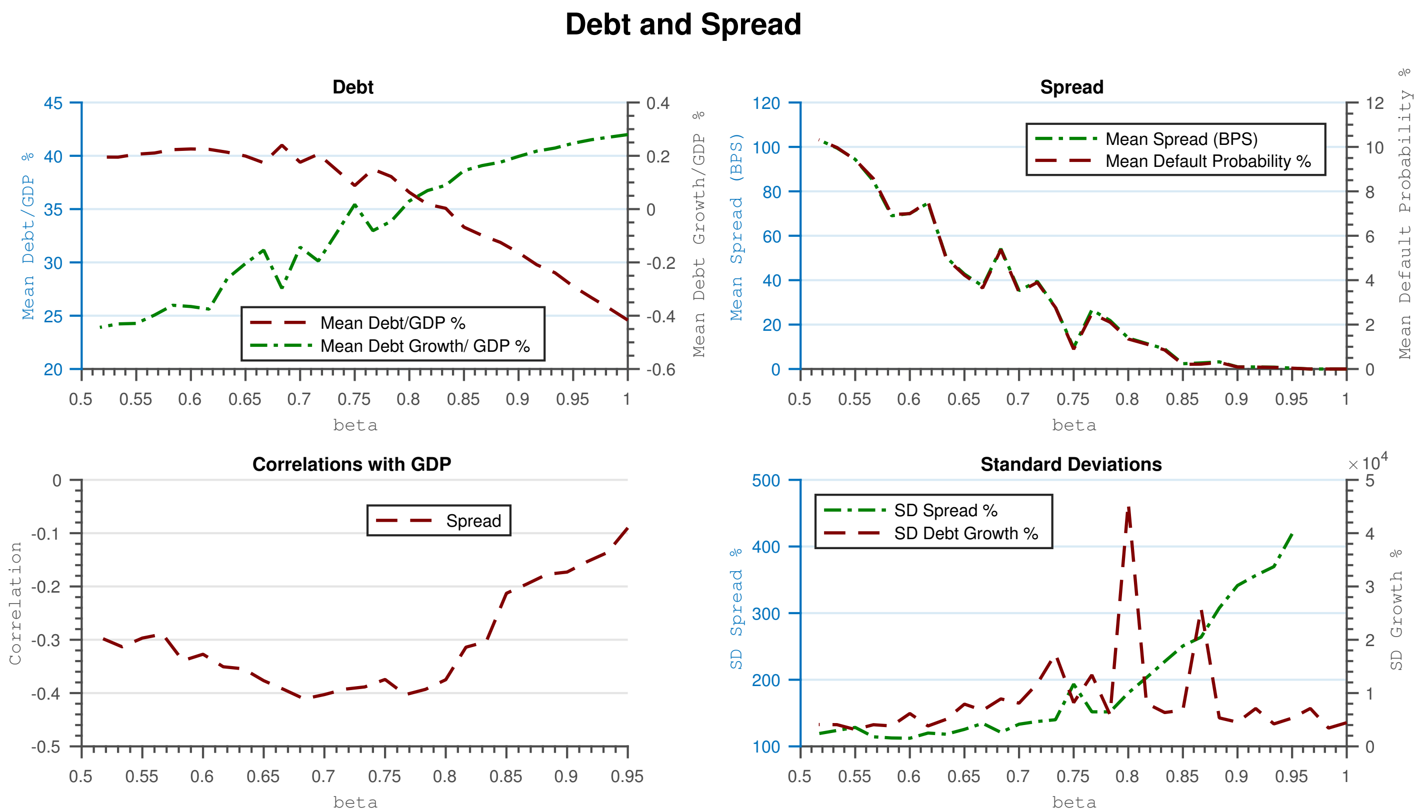

Figure 5: Plots of the summary statistics for different values of β; $\delta = 0.98$. The first 100 quarters are burned, 50 quarters are averaged for 10 000 agents. Summary statistics are obtained averaging over simulations. Standard deviations are rescaled dividing by the mean.

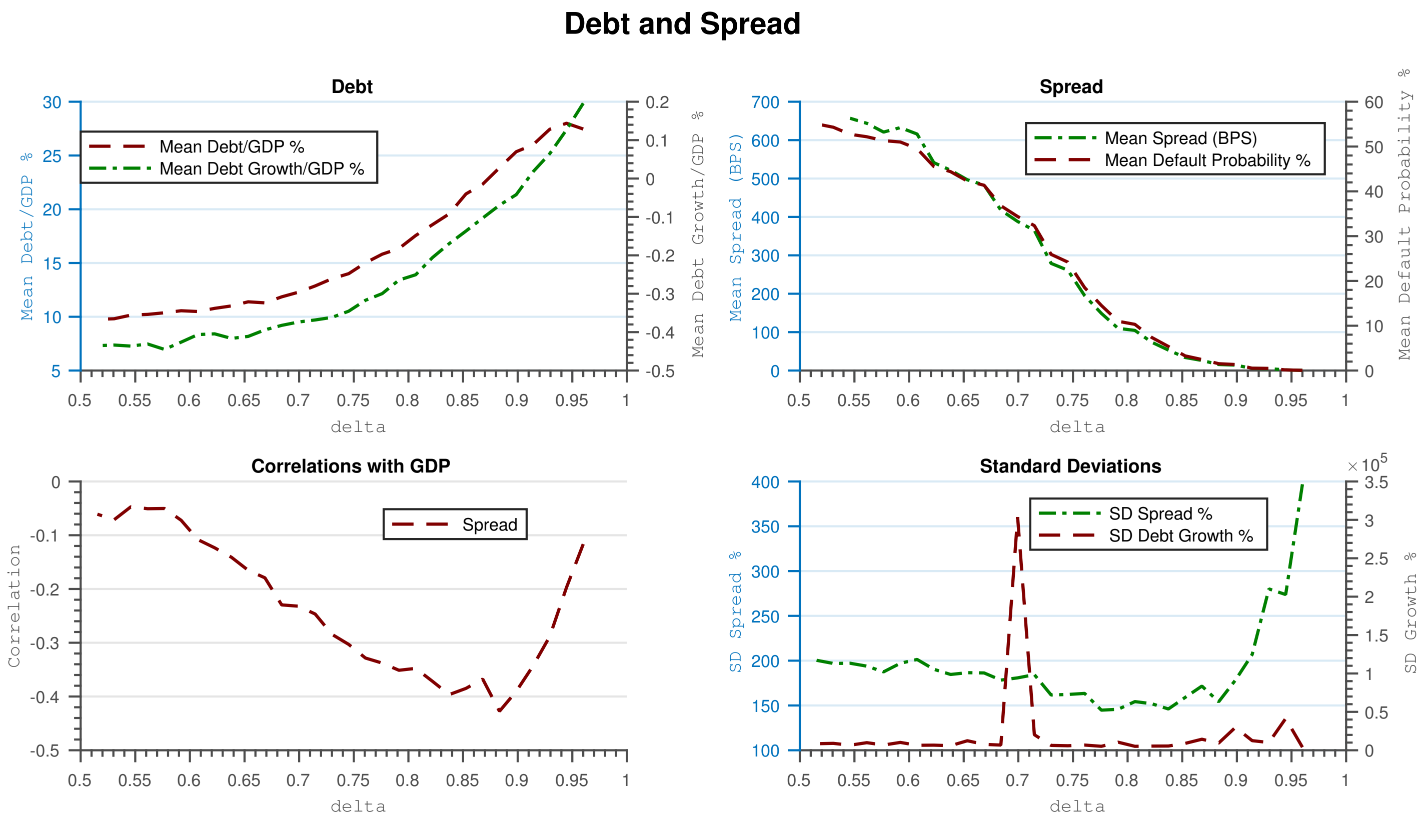

Figure 6: Plots of the summary statistics for different values of δ; $\beta = 1$. The first 100 quarters are burned, 50 quarters are averaged for 10 000 agents. Summary statistics are obtained averaging over simulations. Standard deviations are rescaled dividing by the mean.

Consider also the level of the debt/GDP ratio. It is clear that an additional discounting in terms of immediate gratification raises the level while a higher impatience lowers the level. As argued above, this is a consequence of the incentive to tie the hands of future governments and the pricing mechanism, which rewards stabilization.
It is also interesting to notice that, for both agents, debt growth/GDP ratio in equilibrium[5] is slightly negative (repayment) for high discounting and slightly positive (accumulation) for low discounting. This is intuitive as a patient government can wait longer to increase expenditure and values debt also later in the future.

Figures (7) and (8) show the plots of debt policy functions for different values of the discount factors, conditional on productivity shock values and initial debt. It is possible to see from the policies what mentioned above in greater detail.
First, δ-discounting affects the cyclicality of debt while β-discounting does not. An inpatient government is indeed unstable and debt choice is driven procyclically by shocks' realizations, regardless of the degree of budgetary constriction; a patient but tempted government stabilizes debt and displays an acyclical behaviour when its hands are tied whereas a countercyclical one when able to choose freely.
Second, plots for intermediate initial debt levels prove the argument of debt loops, as on the one hand the tempted agent tries to keep debt at reasonable levels for any productivity shock while an impatient agent falls into the loop as soon as shocks are in the left tail. When initial debt is too high to be repaid for any productivity value, both of the agents will default and act procyclically (or at least acyclically).
Third, this argument can also be extended when conditioning to the productivity shocks. A hyperbolic

[5]Since often in the simulations the level of debt is zero (for instance when entering the market after a default), debt growth is calculated taking first differences. Notice that absolute values of debt accumulation in equilibrium are close to zero (equal to zero for a three digits rounding).

discounter would try to default less and stabilize debt when too high to be repaid, while an exponential discounter would default strategically also when the economic situation is sustainable but initial debt is high and would less often aim for debt stabilization: see for instance the second plot in the left column in Figure (7) compared to Figure (8).

Figures (9) and (10) display the bond price policies for different values of the discounts as a function of productivity shocks conditional on debt levels. Prices are significantly affected by changes in δ, while only when debt is already high additional β-discounting may generate the debt loop. These figures show that bond price is primarily affected by changes in the long-run discount factor, while short-run discount matters only marginally.

To sum up, debt level is primarily influenced by the β-discount, because inconsistency of the preferences creates a scope for self-constraints and debt stabilization. Spread level and volatility are primarily influenced by the δ-discount, because markets consider the entire long-term horizon and punish impatience with high interest rates. In both cases the government will suffer budget constraints, either self-imposed in the form of high initial debt or market-imposed in the form of excessive interest rates.

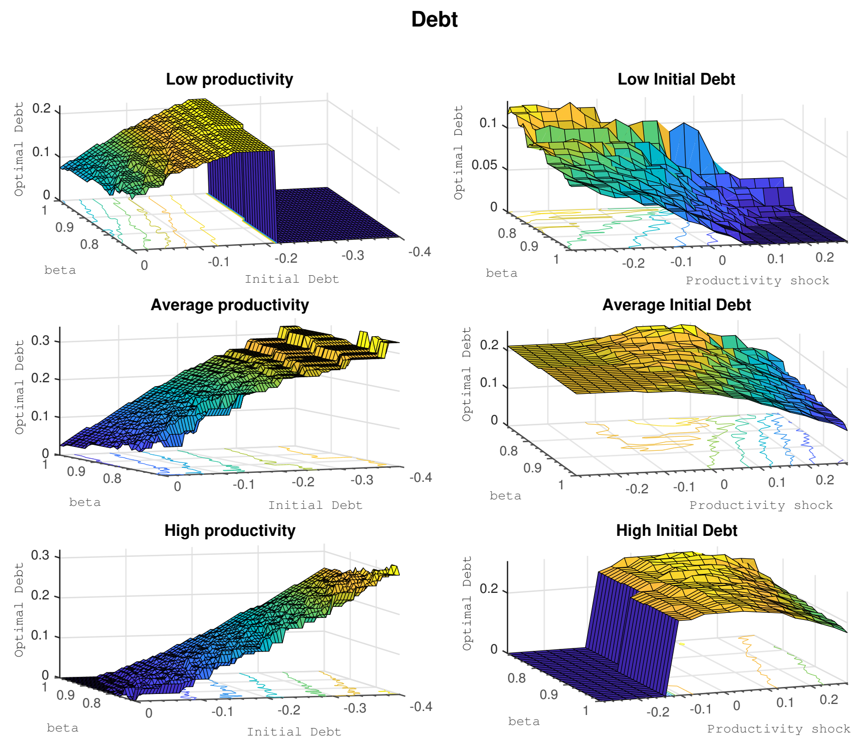

Figure 7: Plots of the debt policy for different values of β; $\delta = 0.98$. Kinks mapping to zero debt correspond to default.

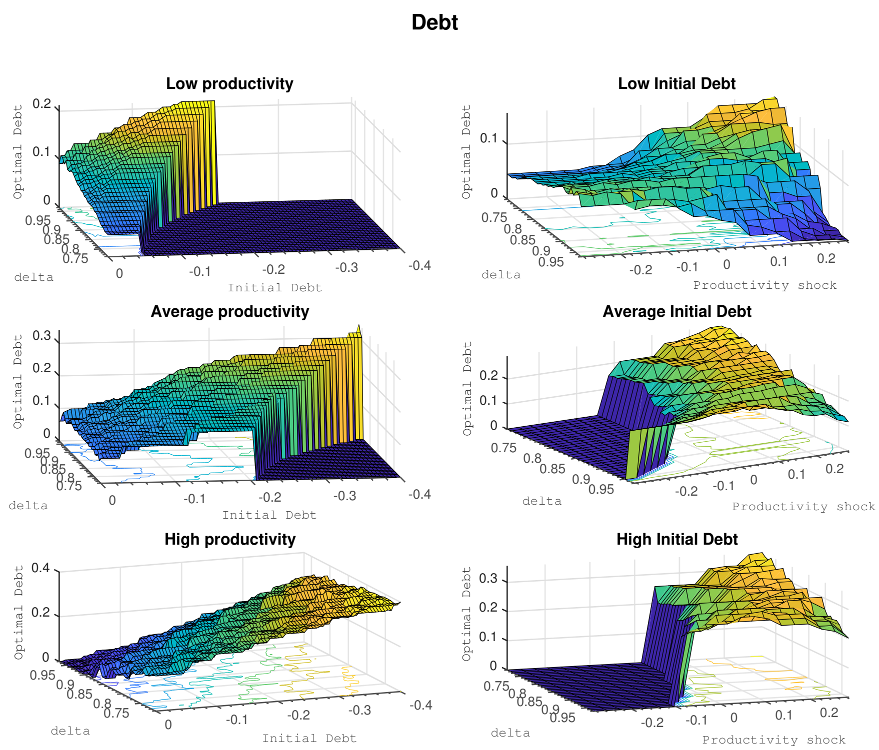

Figure 8: Plots of the bond price for different values of δ as a function of productivity shocks; $\beta = 1$. Value of zero corresponds to default.

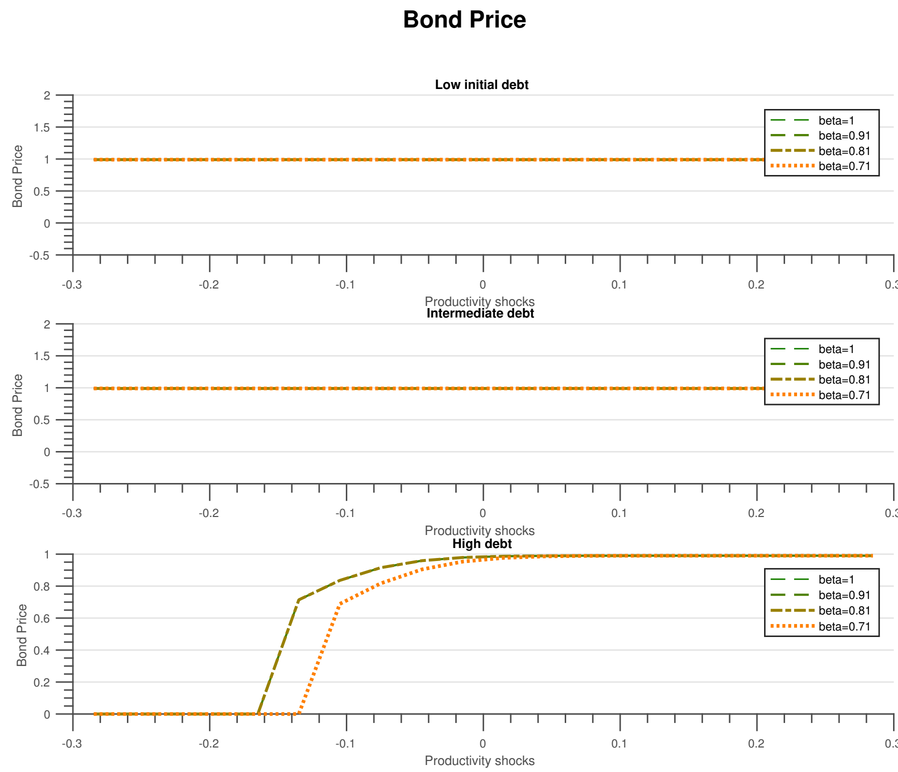

Figure 9: Plots of the bond price for different values of β as a function of productivity shocks; $\delta = 0.98$. Value of zero corresponds to default.

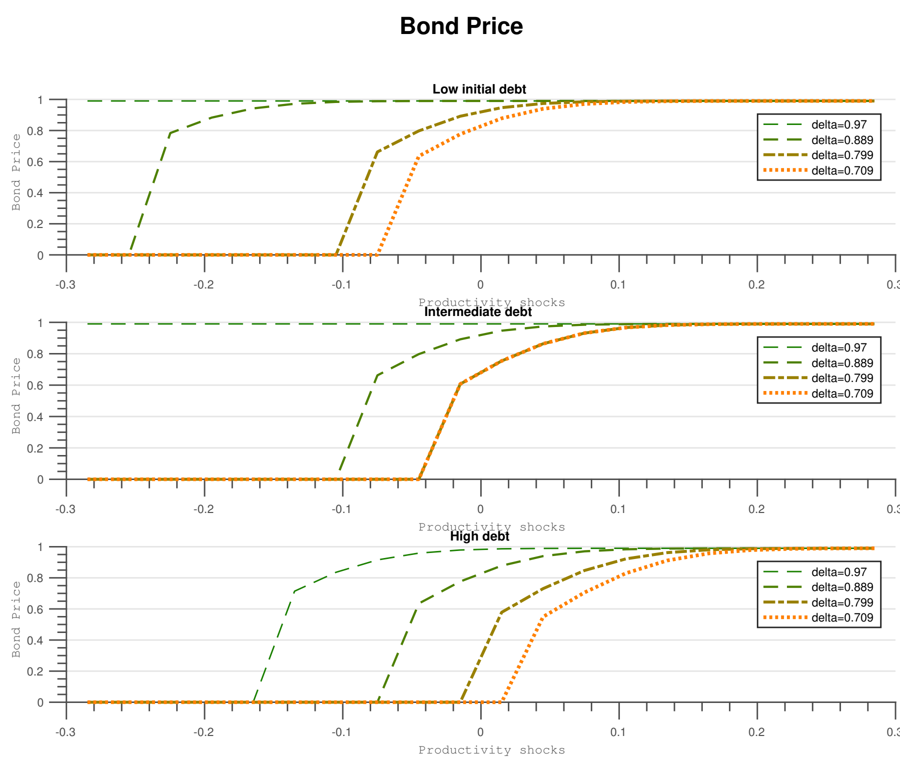

Figure 10: Plots of the bond price for different values of δ as a function of productivity shocks; $\beta = 1$. Value of zero corresponds to default.

5.3 Fiscal policy

Figures (11) and (12) show the summary statistics for taxation, labour and consumption. Correlations are stably close to one for both labour and consumption and their choices are almost invariant to discount rates. However in absolute terms labour moves in the same direction as δ and in the opposite direction as β, that is, impatience translates into shorter working hours while temptation into higher household expenditure and GDP. Consumption is higher when the discount rate for both types is higher. In general debt stabilization is achieved by a large raise in working hours and a small drop in consumption. Similarly the drop in taxation is compensated by a drop in the fiscal expenditure. Therefore the hyperbolic government, once the steady-state level of debt is reached, acts passively to stabilize liabilities and utility smoothing is left to the private initiative. On the contrary the exponential impatient government is active, because in good times keeps the level of debt low and leaves the initiative to private households, while during recessions raises debt often entering in the self-fulfilling default loop, in order to keep consumption high and working hours stable.

The correlation between total governments transfers (public consumption - total taxation) changes significantly with discount rates and moves from negative to positive. Cyclicality of the fiscal policy is therefore a function of the level of discounting, as the more the government anticipates utility the more in equilibrium it would be affected by volatility in shocks, not having saved for bad times. The hyperbolic government tends to act acyclically (slightly procyclically) for $\beta < 0.8$, the exponential procyclically for $\delta < 0.9$. This reflects the intuition on debt cyclicality and the fact that not only a tied tempted government has no room for action but also that severe constraints are market-generated for the impatient executive.
The incentives in play are twofold: on the one hand smoothing utility and, on the other hand, the incentive created by the market forces. Since the higher the productivity shock is, the easier it is to access debt and the lower the price of the bond is, if consuming immediately is prevalent over the smoothing, when GDP is high, then consumption is anticipated. When GDP is low budget constraints always bind and no intervention can be taken to counteract the loss in utility. Therefore the policy turns to be procyclical. The extent to which one effect is prevalent to the other is captured mainly by the elasticity of intertemporal substitution, σ.

The hyperbolic discount factor is in general not able to generate large variations in the fiscal variables, since affecting only the short term horizon. For instance, values in the range (0.75,1) are almost ineffective for all the choice variables: fiscal policy does not change implying a stability also for private consumption and labour supply. On the other hand δ affects significantly real variables as directly modifying the whole stream of future choices, also far in the future.

To sum up, in the stable hyperbolic steady state consumption, taxation and labour are also stable. Also, levels are significantly affected by δ and less by β. The fiscal policy captures fully the cyclicality of government's intervention and helps in disentangling the extent of the budget constraints. Therefore it plays a role in the identification of the time parameters.

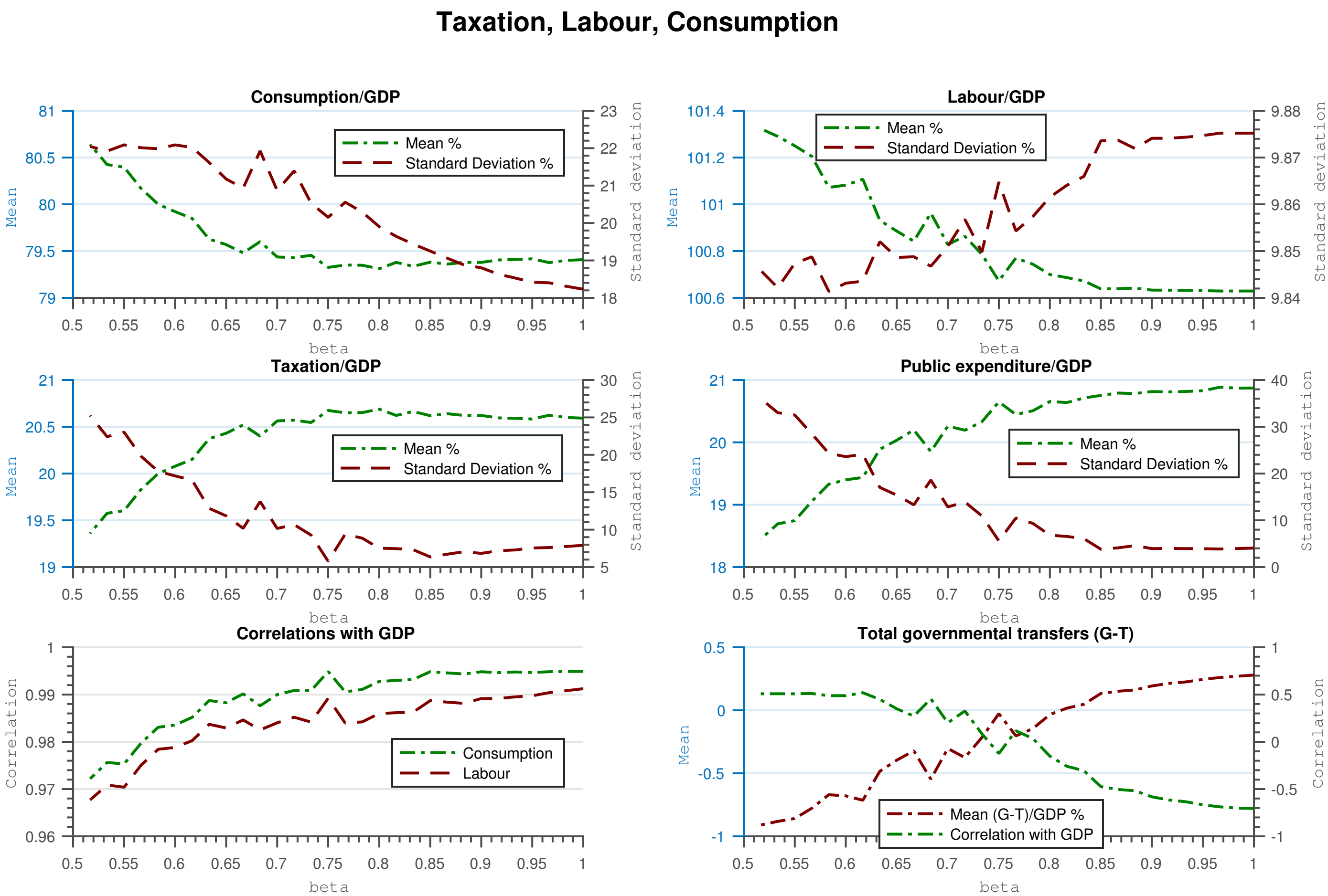

Figure 11: Plots of the summary statistics for different values of β; $\delta = 0.98$. The first 100 quarters are burned, 50 quarters are averaged for 10 000 agents. Summary statistics are obtained averaging over simulations. Standard deviations are rescaled dividing by the mean.

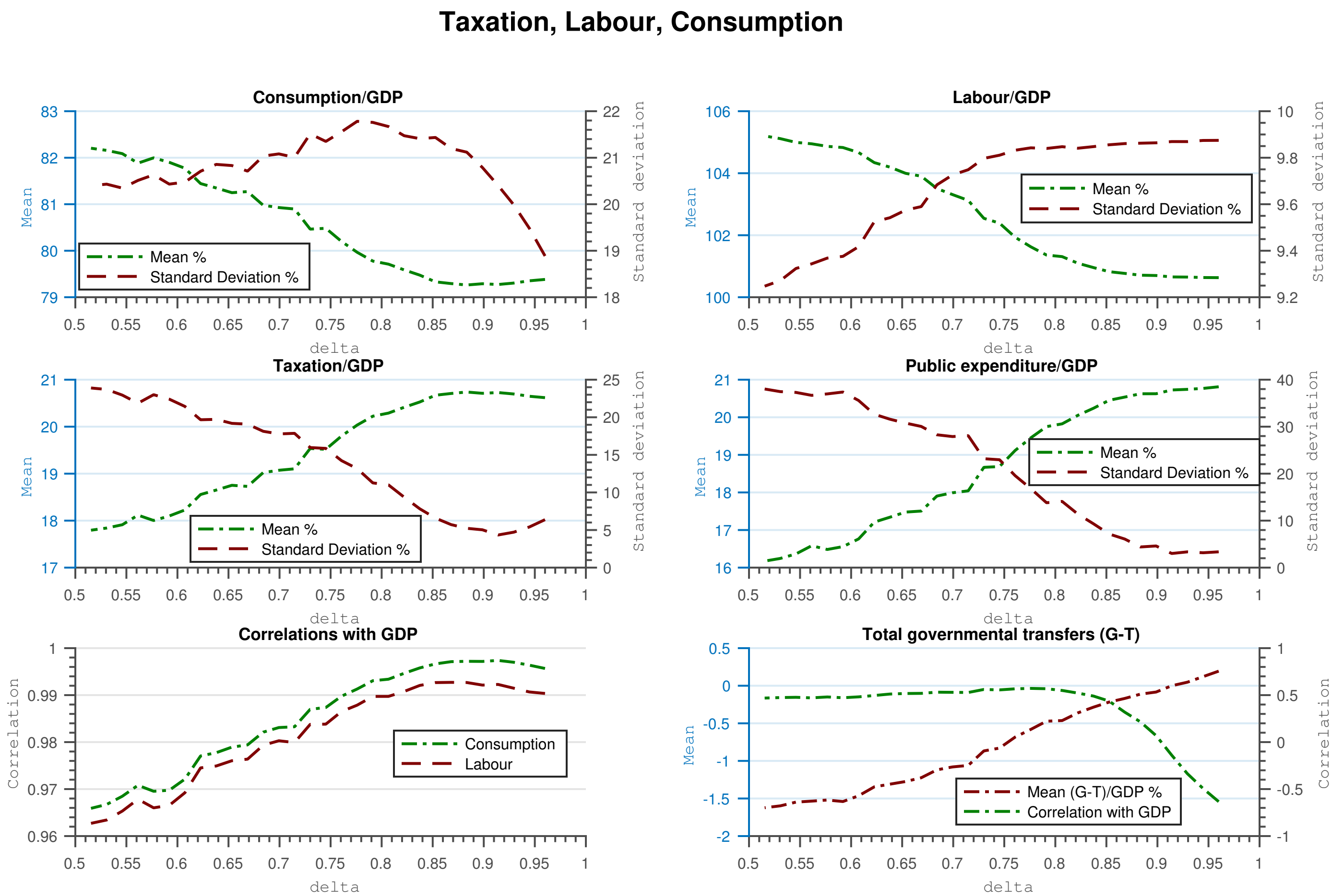

Figure 12: Plots of the summary statistics for different values of δ; $\beta = 1$. The first 100 quarters are burned, 50 quarters are averaged for 10 000 agents. Summary statistics are obtained averaging over simulations. Standard deviations are rescaled dividing by the mean.

5.4 Method of Simulated Moments

The model is calibrated using Italian quarterly data. All the series considered are seasonally adjusted using the Holt-Winters filter. The correlations are calculated on the cyclical component of the filtered series, using the Hodrick-Prescott filter for quarterly data. Table (3) displays the sources of the data used and a short explanation for the variables. Notice that the debt matched is the external/foreign debt, i.e. the debt owned by foreign investors. The spread is calculated over the 3 months discounted bond (minus inflation), prices are obtained in the Datastream database.

Variable	**Item**	**Database**
GDP	Gross domestic product at market prices	Eurostat
C	Final consumption expenditure of households	Eurostat
G	Final consumption expenditure of general government	Eurostat
T	VAT and Taxes on products, production and imports	Eurostat
D	External general government gross debt	World Bank
ZCB 3m Ger	Zero Coupon Bond 3m interest rate for German central government	Thomson Reuters
ZCB 3m Ita	Zero Coupon Bond 3m interest rate for Italian central government	Thomson Reuters
Infl Ger	Inflation Germany	Federal statistical office germany
Infl Ita	Inflation Italy	ISTAT
Spread 3m	Spread between real interest rates (Italy-Germany) at 3m	Calculations

Table 3: Italian data used in the calibration. All the variables are quarterly data. Note: It is possible that this definition of taxes is broader than the one used in the model (tax on consumption only). It can be then interpreted in terms of upper bound for simulation values.

Table (4) and Figures (13)-(14) show the results of the minimization of the loss function matching weighted quadratic errors of the simulations. Errors are calculated as the difference between empirical and simulated moments, the optimal weights used are the precisions (inverse standard deviations) of the simulated moments, assigning lower weights to less precise estimates. As anticipated in the previous section, the mean of debt is primarily affected by β, while the spread is sensitive to changes in δ. When trying to match moments on debt alone or spread alone, the two parameters become almost perfect substitutes and identification is impossible. However, a joint match results in a clear identification scheme.

The addition of other moments on fiscal policy do not significantly change the results: this is not surprising at all since the level of the debt is a function of active policy variables. Nevertheless, it is important to identify the flat area corresponding to values of δ below 0.7. The reason is that real variables in the fiscal policy are severely affected by changes in the long-term discount factor while they are almost neutral to changes in the short-term rate. When δ is low enough, taxation drops and so does public expenditure, much lower than the empirical estimates. On the other hand, when δ is stably close to 0.98 a low β has a small effect and keeps fiscal variables at realistic levels while capturing the appetite for debt and the growth in spreads.

Table (5) presents the summary statistics for the simulations with $\beta = 0.66$ and $\delta = 0.98$. The model is perfectly able to match the actual levels of debt and spread, as well as the sign of the correlation of the spread. However, even this more flexible model struggles to generate large spread volatility without calibrating the income process to non-realistic values. Except for the standard deviation, the model is able to produce realistic values for all the variables. The correlation between GDP and government total transfers displays the wrong sign, because of the tight budget constraints that force a procyclical behaviour in the steady state (but is negative out of the equilibrium).

Moments Matched	Parameters	Values	Moments Matched	Parameters	Values
(1) - Debt			(2) - Spread		
Mean debt/GDP Mean Debt Growth/GDP	beta delta	0.84 0.9	Correlation GDP - Spread Mean Spread (BPS) Standard Deviation Spread	beta delta	0.96 0.54
(3) - Debt and Spread			(4) - Debt, Spread and Fiscal policy		
Mean debt/GDP Correlation GDP - Spread Mean Spread (BPS) Standard Deviation Spread	beta delta	0.66 0.98	Mean debt/GDP Correlation GDP - Spread Mean Spread (BPS) Standard Deviation Spread Mean C/GDP Mean Tax/GDP Mean G/GDP	beta delta	0.52 0.98

Table 4: Moments matched and values of the parameters that minimize the loss function. The period considered to calculate empirical moments is 2011-2017. Standard deviations are rescaled dividing by the absolute value of the mean.

	Moments	Simulation	Data
Correlations	GDP - Spread	-0.363	-0.198
	GDP - Default Probability	-0.036	
	GDP - Government transfers	0.527	-0.263
	GDP - Trade balance	-0.356	
	GDP - Consumption	0.985	0.980
Means	Debt/GDP %	41.34	43.50
	Spread - Basis points	47.50	44.45
	Debt Growth/GDP %	-0.2457	-0.1598
	Default Probability %	4.81	
	TB/GDP %	0.158	
	C/GDP %	79.65	61.62
	T/GDP %	20.34	32.19
	G/GDP %	19.87	19.29
Standard Deviation	Spread	1.182	3.344

Table 5: Simulation results and empirical estimates for the main summary statistics. The period considered to calculate empirical estimates is 2011-2017. Simulations are obtained for $\beta = 0.66$ and $\delta = 0.98$. The first 100 quarters are burned, 50 quarters are averaged for 10 000 agents. Standard deviations are rescaled dividing by the absolute value of the mean.

The figures below show the plots of the loss functions to be minimized. Figure (13) uses the moments on debt/GDP ratio and spread (match (3)). The surface corresponding to low values of δ is almost flat, showing that the two discounts are substitutes. The surface reaches the minimum value at $\beta = 0.66$, $\delta = 0.98$, implying that the agent is patient but highly tempted. The addition to the loss function of moments related to fiscal policy helps in a better evaluation of the flat area (see Figure (14)). The results of the minimization do not significantly change ($\beta = 0.52$, $\delta = 0.98$), but now the flat surface is clearly identifiable than the optimal point. This is a consequence of the fact that taxation drops substantially at low values of the exponential discount factor, while remains stable at low values of the hyperbolic term.

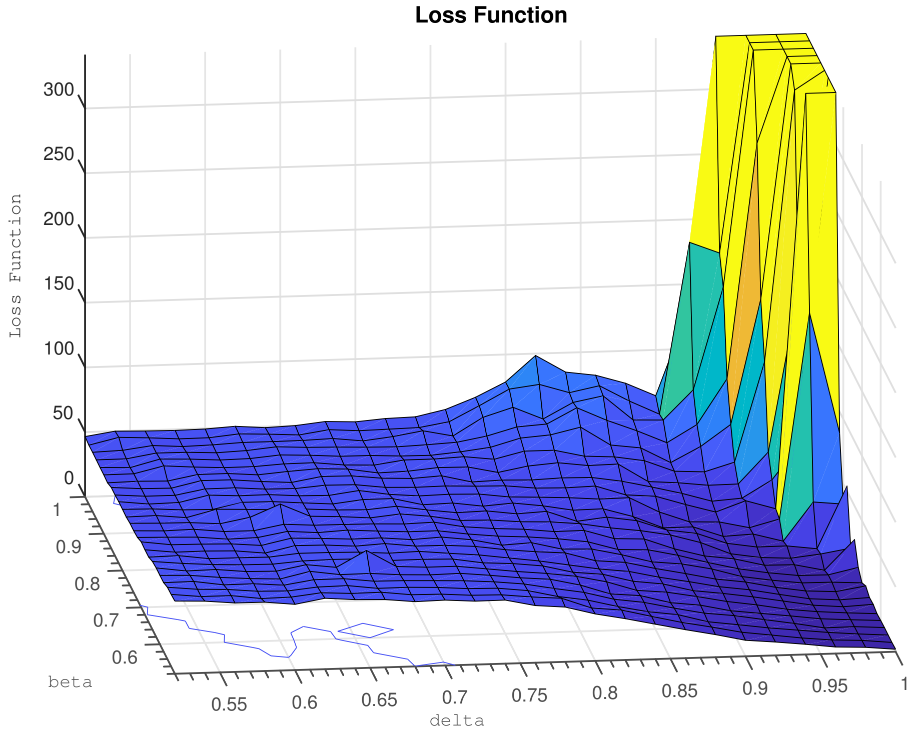

Figure 13: Plots of the loss function in match (3) - Debt and Spread for different values of the parameters. Lowest values correspond to $\beta = 0.66$ and $\delta = 0.98$. The period considered to calculate empirical moments is 2011-2017.

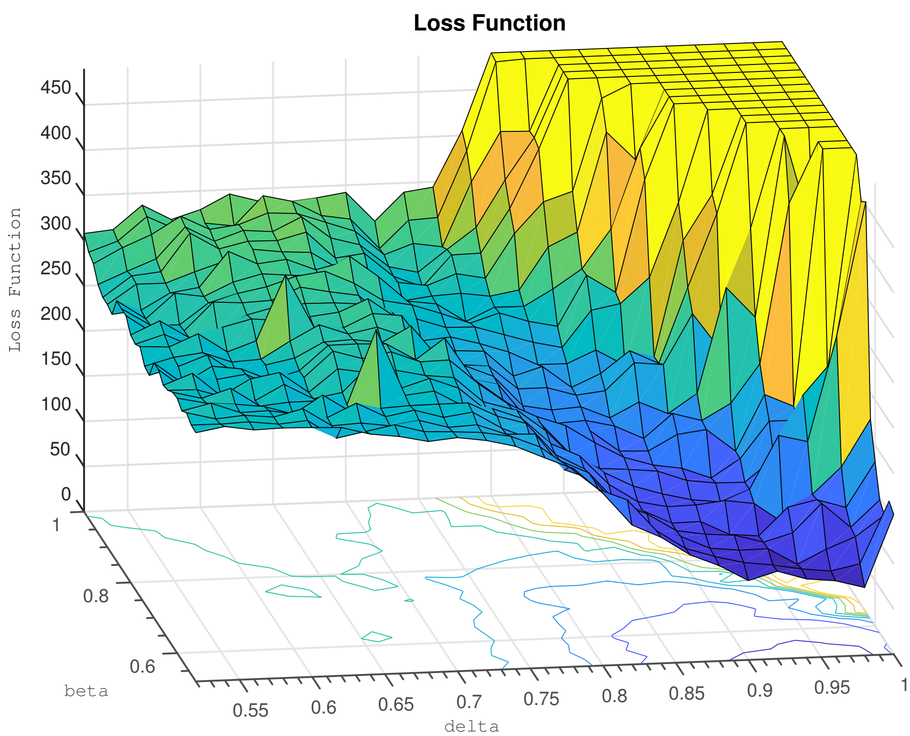

Figure 14: Plots of the loss function in match (4) - Debt, Spread and Fiscal policy for different values of the parameters. Lowest values correspond to $\beta = 0.52$ and $\delta = 0.98$. The period considered to calculate empirical moments is 2011-2017.

6 Conclusions

In this work I developed a sovereign debt model with optimal default and incomplete markets, augmented with fiscal policy and quasi-geometric discounting.

Hyperbolic and exponential discounting are substitutes. However, they are not perfect substitutes and the model is identified. In particular, it is necessary to use the moments on both the spread and the debt level, which have contrasting responses to higher discounting in terms of β and δ. The identification scheme is clear and stable across different matchings: to simultaneously generate high debt ratios and realistic spreads, the government needs to perceive the future patiently, but to be subject to short-term temptation. This preferences imply that, in equilibrium, there is a strong incentive to stabilize debt at high levels, but defaults will not happen often. Markets reward this behaviour with a bond price that is reasonable, given the large liabilities. However, in largely negative economic downturns the government suffers a severe raise in spreads, to counteract the greater incentive to default. This is the main source of variation in spreads, otherwise quite stable; therefore spread volatility is contained. The interpretation is provided in terms of the government "tying the hands" of future administrations, acting patiently to foster patience of the future selves.

The fiscal policy is procyclical as a consequence of the tight constraints imposed by debt repayment; distortionary taxation is high and stable, while public expenditure is lower than taxes, resulting in a debt servicing mostly used in rolling over existing obligations. The household optimally internalises government's choice and in equilibrium bases its utility primarily on variables that it can directly control: private consumption and labour. Both are high and stable, and they are the main source of utility smoothing.

Out of the equilibrium, the story changes. In early periods, when debt has not reached the steady state levels yet, governments offers low taxation and high public expenditure in order to appease citizens' appetites for immediate gratification. In economic upturns households' utility is primarily based on work and private consumption (respectively lower and higher than steady state values), while in downturns the government immediately acts countercyclically, moving the repayment burden to next generations. Debt accumulates steadily until converging to the maximum level, above which the probability of default jumps significantly and the default is triggered. Spread follows the path of debt and grows up to the steady state level. Therefore debt accumulates countercyclically to smooth utility; in addition, large debt bequeaths create a scope for a patient behaviour, through the interaction with lenders. An impatient exponentially discounting government does not have the incentive to stabilization; the only reason not to accumulate enormous amounts of debt is the debt price, which harshly punishes the impatience with large spreads; however due to market incompleteness, the commitment mechanism provided by bond pricing is imperfect. Therefore the response of the current administration aims to keep debt low in economic upturns but issue largely liabilities in downturns, risking default triggers. Hence the debt path is highly unstable and depends on the shock realizations.

Empirical estimates of the summary statistics show that the model including hyperbolic discounting generates realistic values for all the variables of interest. However, even this more flexible model struggles to generate large spread volatility without calibrating the income process to non-realistic values. The estimation of the parameters is obtained using the simulated method of moments, and point estimates are provided. The moments primarily matched are the mean spread and debt/GDP, the volatility of the spread, and the correlation between spread and GDP. In addition, moments on the fiscal policy are added. Benchmark estimates for the full set of moments lead to a high patience (δ = 0.98) and high time-inconsistency (β = 0.52). The values are highly realistic and in line with the existing lab evidence, and suggest that time inconsistency is indeed a plausible explanation for the lack of credibility and the apparent naivety of the Italian institutions when shaping fiscal guidance.

7 Appendix

7.1 Numerical solution

Following [Hatchondo et al., 2009], the model is solved using a backward-induction algorithm, which is run for 250 quarters, generating steady-state policy functions. The number of quarters is chosen to be large enough so have results do not change with additional iterations.

When the time horizon is finite, solving the intrapersonal game with the backward induction algorithm ensures the existence of an equilibrium[6]. That is, given a guess for the policy function in the last iteration T

$$C_T(x_T) = (C_T(Z_T, B_T), G_T(Z_T, B_T))$$

for all $1 \leq t \leq T-1$, the consumption path can be defined as:

$$C_t(x_t) \in argmax_{c \in \mathcal{C}} U(c) + \delta \int_{\mathcal{Z}} (W_{t+1} - \epsilon U \circ g \circ W'_{t+1})(x_{t+1}) d\pi(Z_{t+1}|Z_t) \tag{34}$$

Or in terms of continuation value, given a guess for the continuation value in the last iteration T

$$V_T(x_T) = V_T(Z_T, B_T)$$

for all $1 \leq t \leq T-1$, the consumption path can be defined as:

$$C_t(x_t) \in argmax_{c \in \mathcal{C}} U(c) + \beta\delta \int_{\mathcal{Z}} V_t(x_{t+1}(c, x_t)) d\pi(Z_{t+1}|Z_t) \tag{35}$$

The algorithm is divided in 8 parts:

1. Create the grids for the variables: 20 points in the grid for the income process, 80 for the assets and 50 for the taxes. The income grid is calculated using the method proposed by [Tauchen, 1986]. The asset and tax grids are equally spaced between [-0.4,0] and [0,1] respectively.

2. Use the closed-form results on the choices of the consumer to calculate the utility in default as a function of the tax rate and productivity shocks only (as there is no access to the market). Then the optimal value for the taxation is chosen through a maximization.

3. Calculate the the utility with market access as a function of the tax rate, productivity shocks, initial debt and new debt choice, given an initial guess ($t = T$) for the bond price. Then the optimal value for the taxation is chosen through a maximization.

4. Use the initial guesses ($t = T$) for the continuation and current value function to calculate the updated continuation and value functions, using the utilities found in the previous points. Integrals for the expectations are calculated using the probabilities associated to the grid for the productivity shocks. The optimal values for the new debt are obtained maximizing the current value functions.

5. Compare the realized utility in default with the realized utility with market access and default when the former is higher than the latter. The default rule so obtained will be a two dimensional matrix, function of the productivity shock and the initial level of debt.

6. Update the price of the bond using the default rule and risk-free interest rate.

7. Go back to point (3) using as initial guess the values found in the previous points and iterate backward till $t = 0$.

8. Once the backward induction is finished, the algorithm calls the simulation routine that uses the policy functions for the tax rate, default rule and bond price to calculate the paths for all the variables. The realizations for the productivity and the re-entry shocks are randomly extracted according to the probabilities in the calibration.

[6] See next sections.

7.2 Heuristic derivation of the Generalized Euler Equation

Even under standard properties of the utility function, the assumption of differentiability of the value function, which is key for the derivation of the generalized Euler equation, is hardly trivial. Fortunately, there is some hope of success that builds from established results in the literature. [Harris and Laibson, 2001] prove the existence of a smooth solution in a buffer-stock model without the default option. The generalization from one to multiple state variable (see next section) is straightforward and, conditional on not defaulting, the same results hold. This intuition is exploited to derive -heuristically- a generalized Euler equation under the assumption that market access is permanently forbidden after default ($\lambda = 0$).

The heuristic derivation of the generalized Euler equation follows. Let us denote the partial derivative using a subscript, e.g. $U_{x,t} := \frac{\delta U(C_t, G_t, l_t)}{\delta x_t}$. Denote the Lagrange multipliers associated to the budget constraint with θ_t.

The first order conditions for the choice of taxation, public expenditure and debt are:

T_t:

$$U_{C,t} C^{\star}_{T,t} + U_{l,t} l^{\star}_{T,t} + \theta_t \Big[C^{\star}_{T,t} T^{\star\star}_t + C^{\star}_t \Big] = 0 \tag{36}$$

G_t:

$$U_{G,t} - \theta_t = 0 \tag{37}$$

B_{t+1}:

$$\beta\delta \mathbf{E} \Big\{ V_{B,t+1} \Big| Z_t, D(Z_t, B_t) = 0 \Big\} - \theta_t \left[B^{\star\star}_{t+1} \frac{\delta q_t}{\delta B^{\star\star}_{t+1}} + q_t \right] = 0 \tag{38}$$

where $C^{\star}_{x,t}$ and $l^{\star}_{x,t}$ are the partial derivatives with respect to x of the optimal private consumption and labour responses to the government choices $C(Z_t, T^{\star\star}_t)$ and $l(Z_t, T^{\star\star}_t)$. Notice that in autarky, since $\lambda = 0$, the bond price can take any value without any loss of generality, and the Euler equation is always equal to zero.

If the value functions and the Lagrangian of the problem are continuously differentiable[7], it is possible to take continuous derivatives with respect to B_t of the indirect current value function, plug in the FOCs and apply the Envelope theorem:

$$\begin{aligned} W_{B,t} &= U_{C,t} C^{\star}_{T,t} T^{\star\star}_{B,t} + U_{l,t} l^{\star}_{T,t} T^{\star\star}_{B,t} + U_{G,t} G^{\star\star}_{B,t} + \beta\delta \mathbf{E}_t \Big\{ V_{B,t+1} B^{\star\star}_{B,t+1} \Big\} + \\ &+ \theta_{B,t} \Big[T^{\star\star}_t C^{\star}_t + B_t - q(Z_t, B^{\star\star}_{t+1}) B^{\star\star}_{t+1} - G^{\star\star}_t \Big] + \\ &+ \theta_t \left[T^{\star\star}_{B,t} C^{\star}_t + T^{\star\star}_t C^{\star}_{T,t} T^{\star\star}_{B,t} + 1 - \frac{\delta q_t}{\delta B^{\star\star}_{t+1}} B^{\star\star}_{B,t+1} B^{\star\star}_{t+1} - q_t B^{\star\star}_{B,t+1} - G^{\star\star}_{B,t} \right] \\ &= U_{G,t} \end{aligned} \tag{39}$$

where $T^{\star\star}_{x,t}$ and $G^{\star\star}_{x,t}$ are the partial derivatives with respect to x of the Markov-optimal taxation and expenditure functions $T(Z_t, B_t)$ and $G(Z_t, B_t)$ that maximize the problem in (14) and (21) and $B^{\star\star}_{x,t+1}$ is the partial derivative with respect to x of the Markov-optimal debt choice $B^{\star\star}_{t+1} \equiv B(Z_t, B_t)$ defined such that the equality $q(Z_t, B(Z_t, B_t)) B(Z_t, B_t) = T(Z_t, B_t) C(Z_t, T(Z_t, B_t)) + B_t - G(Z_t, B_t)$ holds. To ease the notation, optimal policies of the government will be denoted without the two stars. One star is kept for the optimal household's responses to remind that have been maximized out in the government's problem.

[7] See next section for a discussion of this assumption.

Taking derivatives of the identity $\beta V_{t+1} \equiv W_{t+1} - (1-\beta)U(C^\star_{t+1}, G_{t+1}, l^\star_{t+1})$ with respect to B_{t+1}:

$$\beta V_{B,t+1} = W_{B,t+1} - (1-\beta)\Big[U_{C,t+1}C^\star_{T,t+1}T_{B,t+1} + U_{l,t+1}l^\star_{T,t+1}T_{B,t+1} + U_{G,t+1}G_{B,t+1}\Big] \tag{40}$$

and plugging equations (39) and (40) into (38) we get the Generalized Euler equation:

$$\begin{aligned} U_{G,t}\left[\frac{\delta q_t}{\delta B_{t+1}}B_{t+1} + q_t\right] = \\ = \delta\mathbf{E}\Big\{U_{G,t+1} - (1-\beta)\Big[G_{B,t+1} + T_{B,t+1}\Big(U_{C,t+1}C^\star_{T,t+1} + U_{l,t+1}l^\star_{T,t+1}\Big)\Big]\Big| Z_t, D_t = 0\Big\} \end{aligned} \tag{41}$$

The Euler equation is interpreted in terms of marginal benefits and costs of additional lending or borrowing. Each unit of additional debt can increase the public consumption today by $\frac{\delta q_t}{\delta B_{t+1}}B_{t+1} + q_t$, which reflects the interest payment and the change in price due to the higher riskiness. The derivative $\frac{\delta q_t}{\delta B_{t+1}}B_{t+1}$ is intuitively negative, implying that the government obtains less from borrowing than it would have obtained without the market contractions.

The RHS of the Generalized Euler equation equals the marginal utility of public consumption at $t+1$ in the case of $\beta = 1$. When $\beta < 1$, there is a higher incentive for raising the consumption today. That incentive is in turn a function of the responsiveness of public and private variables to a raise in the debt level. First, it depends on the way B_{t+1} affects the public expenditure in $t+1$. Second, it depends on how the new liabilities impact taxation reflecting to private labour and consumption choices.

An additional interpretation can be proposed in terms of liquidity constraints. The more the government is liquidity constrained, the more it would be willing to raise public expenditure out of new debt, hence increasing expenditure right away. But if the budget constraint is binding, new liabilities are accompanied by higher taxes, which will distort consumption and taxation differently depending on the value of the parameters.

Given the functional form for the utility in (2), the term $U_{C,t+1}C^\star_{T,t+1} + U_{l,t+1}l^\star_{T,t+1}$ equals to:

$$\pi\left[\frac{\left(\frac{Z_{t+1}}{1+T_{t+1}}\right)^{\frac{1+\omega}{\omega}}}{\frac{1+\omega}{\omega}}\right]^{1-\sigma}\left[\frac{1}{1+T_{t+1}}\right]\left(\frac{1+\omega}{\omega^2} - 1\right)$$

and therefore depends on the parameters ω and σ as well as the level of taxation and of the productivity shocks and will be positive if and only if $\omega \in [-0.62, 1.62]$, that is the Frisch elasticity is below 0.62.

7.3 Smoothness of the Value Function

[Harris and Laibson, 2001] discuss, in the discrete-time case, the properties of the value function and of the equilibrium. To ease notation, assume that:

$$\mathbf{E}_t\left[U(C_t) + \beta\sum_{i=1}^{\infty}\delta^i U(C_{t+i})\right] \tag{42}$$

where $\beta \in [0,1]$ and $U : [0,+\infty) \to [-\infty,+\infty)$. The model is equivalent to the standard exponential discounted game with $\beta = 1$ and this version can be seen as a perturbation in a "nearby" game.

In a recursive notation, let $W(x_t)$ be the current value function of state of the economy $x_t := (Z_t, B_t)$ in t and $V(x_t)$ the continuation value function form the perspective of self t. Let $C_t := f(C_t, G_t)$ be and equilibrium consumption function that represents the optimal choice given the states of the economy x_t at time t and aggregates together private and public consumption under a unique notation. Adopt the perspective of self t. Then from her perspective the continuation value is not biased by the quasi-geometric discounting β, implying that:

$$V(x_{t+1}) = Y(C(x_{t+1})) + \mathbf{E}_{t+1}[\delta V(x_{t+2})] \tag{43}$$

Notice that the states of the economy in $t+2$, i.e. x_{t+2} is a function of the choice of consumption in $t+1$: $x_{t+2}(C(x_{t+1}), x_{t+1})$. $V(x_{t+1})$ is the expectation, conditional on x_{t+1}, of the present discounted value of the utility stream which starts in period $t+1$.

However at time t the current self would use a different discount factor $\beta\delta$. Therefore the current value function solves the equation:

$$W(x_t) = U(C(x_t)) + \mathbf{E}_t[\beta\delta V(x_{t+1})] \tag{44}$$

where again x_{t+1} is a function of the choice of consumption in t: $x_{t+1}(C(x_t), x_t)$.

The optimal choice for consumption is taken with respect to this last equation, as it is decided by the current self:

$$C(x_t) \in \mathit{argmax}_{c\in\mathcal{C}} U(c) + \mathbf{E}_t[\beta\delta V(x_{t+1}(c, x_t))] \tag{45}$$

This implies that the Euler equation has a generalized form including the discount factor and an additional term giving higher weight to future states of the economy (as perceived very valuable before temptation to increase disproportionately consumption today kicks in). Moreover the trade-offs given by the FOCs of the maximization problem of the consumer change along the evaluation horizon.

The first order condition associated with (45) implies that:

$$U'(C(x_t)) \geq \beta\delta\mathbf{E}_t\left[\frac{\delta V(x_{t+1}(C(x_t), x_t))}{\delta x_t}\frac{\delta x_{t+1}(C(x_t), x_t))}{\delta x_t}\right] \tag{46}$$

with equality when $C(x_t) \in \mathcal{C}(x_t)$ is an interior solution.

Using the first order condition and the envelope theorem, the shadow value of the state of the economy equals the marginal utility of consumption as a function of the state.

$$\frac{\delta W(x_t)}{\delta x_t} = \frac{\delta U(C(x_t))}{\delta x_t} \tag{47}$$

By definition V and W are linked in the equality:

$$\beta V(x_{t+1}) = W(x_{t+1}) - (1-\beta)U(C(x_{t+1})) \tag{48}$$

and combining (48) with (47) we get the Generalized Euler equation for the hyperbolic-discounter self t:

$$\begin{aligned} U'(C(x_t)) &\geq \beta\delta\mathbf{E}_t\left[\frac{\delta V(x_{t+1}(C(x_t), x_t))}{\delta x_t}\frac{\delta x_{t+1}(C(x_t), x_t))}{\delta x_t}\right] \\ &= \delta\mathbf{E}_t\left[\left(\frac{\delta W(x_{t+1})}{\delta x_t} - (1-\beta)\frac{\delta U(C(x_{t+1}))}{\delta x_t}\frac{\delta C(x_{t+1})}{\delta x_t}\right)\frac{\delta x_{t+1}}{\delta x_t}\right] \\ \Rightarrow U'(C(x_t)) &\geq \mathbf{E}_t\left[U'(C(x_{t+1})\left[\beta\delta C'(x_{t+1}) + \delta(1 - C'(x_{t+1}))\right]\frac{\delta x_{t+1}}{\delta x_t})\right] \end{aligned} \tag{49}$$

If $\beta = 1$ the relation simplifies to the standard Euler equation for the exponential case:

$$U'(C(x_t)) \geq \mathbf{E}_t\left[\delta U'(C(x_{t+1}))\frac{\delta x_{t+1}}{\delta x_t}\right] \tag{50}$$

Recursive approach requires existence and uniqueness results to be safely applied, which can be provided also in this more general case, with some modifications. Equation (44) can be rewritten as

$$W_t(x_t) = \max_{c\in\mathcal{C}} U(c) + \beta\delta\int_{\mathcal{Z}} V_t(x_{t+1})d\pi(Z_{t+1}|Z_t) \tag{51}$$

Therefore using the equivalence (48):

$$W_t(x_t) = \max_{c \in \mathcal{C}} U(c) + \delta \int_{\mathcal{Z}} (W_{t+1} - (1-\beta) U \circ C_{t+1})(x_{t+1}) d\pi(Z_{t+1}|Z_t) \tag{52}$$

Substituting from equation (47) we get

$$\begin{aligned} W_t(x_t) &= \max_{c \in \mathcal{C}} U(c) + \delta \int_{\mathcal{Z}} (W_{t+1} - \epsilon U \circ g \circ W'_{t+1})(x_{t+1}) d\pi(Z_{t+1}|Z_t) \\ &= (\mathcal{B} W_{t+1})(x_t) \end{aligned} \tag{53}$$

where $\epsilon := (1-\beta)$ and $g := (U')^{-1}$

Equation (53) is the Bellman equation of the hyperbolic discounter and $\mathcal{B}$ is the Bellman operator. The smoothness results on the value function follow from the next proposition. Proofs can be found in [Harris and Laibson, 2001].

Proposition 7.1 *Let $\mathcal{BV}^1_{loc}$ be the space of equivalence classes of functions of locally bounded variation*[8] *such that if $W \in \mathcal{BV}^1_{loc}$ then also $W' \in \mathcal{BV}^1_{loc}$ and suppose that $W : (0, +\infty) \to \mathbf{R}$. Then $W \in \mathcal{BV}^1_{loc}((0, +\infty))$ if and only if W is the difference of two convex functions.*

The result proves that the Bellman operator $\mathcal{B}$ is a self-map on the space $\mathcal{BV}^1_{loc}$, hence implying existence of a solution. This proposition can also be extended to the infinite-horizon case. Being a self-map on the "local" space $\mathcal{BV}^1_{loc}$ is also the basis of the main result that ensures convergence of value function iteration and finite-horizon equilibrium selection.

Theorem 7.2 *Suppose that $\delta \frac{\delta x_{t+1}}{\delta x_t} < 1$ for every t and that U is three times continuously differentiable on its support. Then for every appropriate initial condition x_0 there exists a range of values of β near 1 such that for every β in that range, the equilibrium exists and is unique.*

The theorem proves a local uniqueness result. With that tool it is possible to obtain the unique local solution of any game with hyperbolic discounting for small variations from the standard exponential solution. Locally some of the main regularities are saved also in the hyperbolically-discounted problem.

7.4 Uniqueness of the solution for the "nearby" game

Further insights on the topic can be found in [Judd, 2004]. The following approach can be applied to the results in [Auclert and Rognlie, 2016], showing the uniqueness of the equilibrium in the standard setting with permanent exclusion from markets ($\lambda = 0$) and Markov income process[9]. Judd presents a theoretical constructive approach to solve problems with hyperbolic discounting and, more generally, any problem that can be described as a perturbation of a game that can be solved trough value function iteration -i.e. "nearby" games-.

The substantial difference between Judd's work and the previous section is that Harris and Laibson's result assumes income uncertainty, critical in smoothing out their problem and avoid mathematical difficulties. Judd on the contrary takes an approach that is valid also for the deterministic case. In addition Harris and Laibson prove only that a set of solutions is a semicontinuous correspondence in hyperbolic discounting, whereas Judd constructs a manifold of solutions, one for each value of hyperbolic discounting. Also Judd uses ingredients from calculus and applying the same constructive approach that is the base of the Banach contraction mapping theorem.

[8] Locally bounded variation refers to the fact that there exist two increasing functions such that the bounded function is the difference of those.

[9] A formal proof of the claim is beyond the scope of the research.

The starting point is to define a general operator equation

$$G(x_t, x_{t+1}, \epsilon C'(x_{t+1}), \epsilon) = 0 \tag{54}$$

where $\epsilon = \frac{1}{\beta} - 1$, that rewrites the Generalized Euler equation in (49).

Trivially when $\epsilon = 0$ it is the exponential case.

The parameter ϵ appears in two distinct places in the equation G, as a part of the discounting term and multiplying the derivative of the policy function with respect to the state variable at time t.

All the issues arise from the fact that changing ϵ changes the nature of the operator equation with a formal perturbation.

Ideally we would like to apply usual perturbation techniques as the Taylor expansion to deal with the problem. Luckily some local results apply and help in doing this.

Let $N(C, \epsilon)(x)$ be defined as:

$$N(C, \epsilon)(x) := G(x_t, x_{t+1}, \epsilon C'(x_{t+1}), \epsilon) \tag{55}$$

We are looking for the solution of the form

$$\begin{aligned} C(x, \epsilon) = C(\bar{x}) + \epsilon \frac{\delta C}{\delta \epsilon}(x, 0) + \frac{\epsilon^2}{2} \frac{\delta^2 C}{\delta \epsilon^2}(k, 0) + \dots \\ s.t.\ \ N(C(x, \epsilon), \epsilon)(x) = 0\ \forall \epsilon \end{aligned} \tag{56}$$

Taking derivatives of $N(C(x, \epsilon)$ in equation (55) with respect to ϵ we get:

$$N_C(\bar{C}, 0) C_\epsilon + N_\epsilon(\bar{C}, 0) = 0 \tag{57}$$

But C_ϵ is the only ingredient in (56) that we do not know; clearly it is possible to obtain it only in the case that $N_C(\bar{C}, 0)$ is invertible.

The next result will provide the conditions under which this property holds. The proof can be found in [Judd, 2004].

Corollary 7.3 *Let C^{m-1} be the space of continuously differentiable functions of order $m-1$ and $||.||_m$ defined by*

$$||\mathbf{f}||_m := \max_{0 \le i \le m} \sup_{xinU} ||\mathbf{D^i f(x)}||$$

where D^i is the i-th derivative operator.

$N_C(\bar{C}, 0) : \mathcal{C} \rightarrow C^{m-1}(I, \mathbf{R})$ is an invertible C^{m-1} operator if and only if value function iteration in the associated problem with $\epsilon = 0$ is locally convergent in the C^{m-1} topology.

This result ensures in a single elegant corollary that if the value function iteration in a dynamic programming problem converges to a unique solution, then there exists a unique equilibrium for "nearby" games and time iteration is a convergent algorithm.

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