Liquidity and Fundamental Risks in a Search Economy

Luca Gagliardone

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Abstract

This paper studies the interplay between liquidity and fundamental risks in an asset pricing framework with a frictional, decentralized secondary market and endogenous trading decisions. In this setting, the liquidity value of assets decreases in the riskiness of the underlying. For a sufficiently large deterioration of fundamentals, agents stop trading the asset, leading to a freeze of the secondary market and flight-to-safety behavior. This mechanism implies a novel type of monetary "safe-trade" equilibrium, in which assets are traded if and only if safe. Liquidity feeds back into the default decision of the issuing firm, potentially leading to price spirals and a multiplicity of equity valuations and default thresholds.

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* New York University. Email: luca.gagliardone@nyu.edu; I thank Jaroslav Borovicka, Man Chon Iao, Julian Kozlowski, Ricardo Lagos, Erik Madsen, Guido Menzio, Gabriel Toledo, Shengxing Zhang, and seminar participants at various institutions for helpful comments and conversations.

1 Introduction

Liquidity and fundamental risks affect a significant portion of credit spreads for several asset classes in financial markets. Whereas fundamental risk originates from exogenous fluctuations in the profitability of debt-financed investments, liquidity risk reflects the uncertainty regarding agents' endogenous decisions to re-trade assets in secondary markets.

These two sources of risk premia are tightly related. In fact, a large body of empirical work has shown that the liquidity of an asset tends to dry up when its profitability drops.[1] When fundamentals deteriorate significantly leading to higher riskiness for debtholders, financial agents are often unwilling to accept the asset as a means of exchange or collateral in trade contracts, effectively resulting in a decrease in the asset's liquidity value. With large and sudden aggregate fluctuations, this mechanism can lead to market freezes and flight-to-safety behavior.[2]

To generate these empirical regularities, this paper incorporates search frictions in an otherwise standard asset pricing framework. A productive risky project is financed by issuing bonds in a primary market. As in Leland (1994) and Leland and Toft (1996), the issuing firm operates under limited liability and can default on its debt at any moment. Workers purchase the bond in the primary market and gather in a secondary decentralized frictional market to trade their production (commodities). As in Diamond (1984), Kiyotaki and Wright (1993), and Lagos and Rocheteau (2009), the bond can be used as a medium of payment for transactions in the secondary market, with workers exchanging commodities for the financial asset.

A key feature of the model is that trading decisions in the secondary market are endogenous and thus depend on the riskiness of the asset. This dependency leads to a novel type of monetary equilibrium in which the bond is accepted as a means of exchange in the secondary market if and only if it is safe. This "safe-trade equilibrium" is characterized by a unique trading threshold for the

[1] See for example Edwards et al. (2007), Bao et al. (2011), Dick-Nielsen et al. (2012b), Friewald et al. (2012), and Chen et al. (2018).

[2] See for example Acharya et al. (2011), Næs et al. (2011), and De Santis (2014).

profitability of the firm's project: when the fundamental is above the threshold, i.e. the bond is distant from its default value, workers accept the asset in exchange for commodities; however, as soon as the cash flow deteriorates too much and touches the trading threshold, the market freezes.

For sensible parameterizations of the coupon rate and production cost of the commodity, this monetary equilibrium may coexist with a non-monetary equilibrium in which the asset is never traded. The difference between the asset prices in the two equilibria provides an intuitive notion of liquidity value, which decreases monotonically in the probability that the firm defaults on its debt. Moreover, as search frictions of the secondary market reduce, the price in the monetary equilibrium goes up, leading to an increase in the liquidity value of the bond.

When the issuing firm faces rollover risk (He and Xiong 2012c), the multiplicity of equilibria in the secondary market generates a multiplicity of equity valuations and liquidation thresholds for the firm. In particular, when financial agents correctly anticipate that the bond will not be traded in the secondary market, the issuing price in the primary market does not include a liquidity value, and thus the firm will suffer larger rollover losses or smaller gains. This channel reduces the equity valuation and implies that the firm defaults on its liabilities more often than in the safe-trade equilibrium.

Additionally, debt rollover generates a "liquidity-default loop," which is also present in the class of models with search frictions discussed in He and Milbradt (2014). An exogenous worsening of the search frictions not only lowers the price in the secondary market but also feeds back into the default decision of the firm through the reduction in equity value. With lower equity value, the firm defaults more frequently, and this further lowers asset prices, leading to a price spiral.

The paper develops as follows. Section 2 describes the model. Section 3 characterizes the trading decisions. Sections 4 and 5 characterize the equilibria of secondary and primary markets. Section 6 discusses extensions and applications. Section 7 concludes.

1.1 Previous Literature

This paper contributes to three strands of the literature.

First, the model extends seminal work by Diamond (1984), Kiyotaki and Wright (1993), and Lagos and Rocheteau (2009) to allow for fundamental risk and shows how this interacts with the liquidity value of the means of exchanges. Additionally, the bond has an intrinsic value due to the coupon payments and thus is related to capital as a competing means of exchange, as in Lagos and Rocheteau (2008), and higher-return assets, as in Hu and Rocheteau (2013).

Second, the paper contributes to the discussion of the asset pricing implications of liquidity. Seminal contributions of this literature are Amihud and Mendelson (1986), Amihud et al. (2006), He and Xiong (2012a), and He and Xiong (2012c). Differently from those works, the liquidity premium is here endogenous and depends on the search frictions of the secondary market. Complementary to this paper, He and Milbradt (2014) develops an asset pricing framework for over-the-counter markets with search frictions as in Duffie et al. (2005). Besides differences in the setting, in that paper trading decisions in the secondary market are exogenous and thus do not depend on fundamental risk. Other related papers in that literature are Duffie et al. (2007), Garleanu and Pedersen (2007), and Lagos (2010).

Finally, this paper contains insights that apply broadly to several applications discussed in the literature. First, it can generate endogenous market freezes as in Acharya et al. (2011) and Gu et al. (2021). Second, it can rationalize flight-to-liquidity/safety behaviors as those discussed in Longstaff (2002), De Santis (2014), and Baele et al. (2020). Third, the model can be interpreted in the context of the US treasury market to interpret empirical evidence such as the negative correlation between AAA-Treasury spread and debt-to-GDP ratio (Krishnamurthy and Vissing-Jorgensen 2012), or business-cycle fluctuations in liquidity premia (Chaumont 2020). In particular, the latter paper develops a model for the US treasury market with search frictions and optimal defaults as in the sovereign debt literature built on the Eaton-Gersovitz framework. In contrast, this paper models default as in the asset pricing literature, which leads to full tractability and closed-form solutions.

2 The Model

The economy is populated by investors and workers. Investors own the shares of a representative firm and receive the firm's cash flow. The firm owns a productive risky project, which is financed by issuing coupon bonds. Bonds are sold to workers on the primary market. Workers produce and consume commodities, and exchange their production and bonds in a decentralized secondary market with search frictions.

Firm's Fundamental. The productive risky project generates an (after-tax) cash flow at rate $V_t > 0$, where $\{V_t : 0 \leq t \leq \infty\}$ follows a geometric Brownian motion under the risk-neutral probability:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dZ_t \qquad \text{(Fundamental)}$$

with μ the constant growth rate of the cash flow, σ the constant volatility, and $\{Z_t, \mathcal{F}_t : 0 \leq t < \infty\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. I assume that a default-free asset exists and it pays continuously an interest rate $r > \mu$.

Defaultable Coupon Bonds. The firm finances the project by issuing a measure $0 < M < 1$ of bonds, which continuously pays a fixed coupon rate $c > 0$ unless the firm declares bankruptcy. Bankruptcy is modeled as an absorbing barrier V_b for the cash flow V_t. If bankruptcy occurs, assets are liquidated and a fraction $0 \leq \alpha \leq 1$ of the value will be lost, leaving debtholders with the present value of the productive asset minus the liquidation cost $\frac{(1-\alpha)}{r-\mu}V_b$. Bonds mature stochastically at rate $1/T$, where T is the expected maturity of bonds. Denote the principal of the bond with p.

Secondary Market The secondary market is only accessible to infinitely-lived workers, that produce consumption goods (commodities), hold bonds, trade bilaterally with other workers, and consume. Commodities and bonds are indivisible, and each agent can hold at any time at most one unit.

The secondary market is incomplete and frictional as in Diamond (1984) and Kiyotaki and Wright (1993). Specifically, there is a continuum of varieties of perishable commodities, produced by workers upon paying a unitary cost. Workers do not consume their own production but search for another agent to trade it, and consume only when trading occurs. Upon meeting, if both workers are willing to exchange their production for the other commodity (double coincidence), they produce their own variety at the unitary cost, then commodities are directly exchanged and consumed. Alternatively, if no agent is willing to exchange (no coincidence), production and trades do not take place. When only one worker wants to trade commodities but the other does not (single coincidence), bond holdings may become relevant as a media of exchange. The counterparty that wants to trade, when endowed with a bond, has to option to offer it in exchange for the commodity. The other agent can then accept the trade, therefore pay the cost, produce the commodity, and exchange it for the bond, or reject it.

Let us normalize the measure of workers to one. A fraction M of workers initially holds a bond and enjoys a flow utility c (coupon rate) from holding it. Bilateral meetings are Poisson events that occur at the rate β; conditionally on a meeting, double coincidence meetings happen at rate δ, single coincidences at rate θ, and no coincidences at rate $1-\delta-2\theta$. The unitary production cost is $\kappa \geq \frac{(1-\alpha)}{r-\mu} V_b$, and consumption generates utility $u > \kappa$. The bond price in the secondary market is denoted by P.

Stationary Debt Structure. Following Leland (1994) and Leland and Toft (1996), assume that the firm commits to replace maturing bonds with newly issued ones of identical face value so that the debt structure of the firm is constant over time.[3] New issuances are purchased by workers on the primary market at the market price $\tilde{P}$. Differences between the maturing bond price and the principal $\tilde{P}-p$ generate rollover losses or gains for the firm.

[3]The assumption of commitment implies that debt rollover does not generate time inconsistency in the default decision. See DeMarzo and He (2021) for a setting that relaxes this assumption.

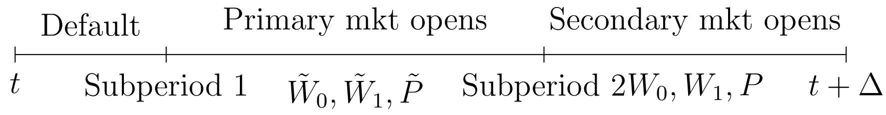


Figure 1: Timing of the model.

Equity Valuation and Endogenous Default. Following He and Xiong (2012c), assume that the firm's rollover gains will be immediately paid out and losses immediately absorbed by issuing new equity. Therefore, the net cash flow rate NC–continuously paid to investors–corresponds to

$$NC(V) := V - \zeta c + \frac{M}{T}\left(\tilde{P}(V) - p\right) \qquad \text{(Cash flow)}$$

where ζ/M is the per-coupon payment net of tax benefits. The value of equity E is the discounted expected value of the firm's future cash flows, given by:

$$E_t := \mathbb{E}_t\left[\int_t^{\infty} e^{-r(s-t)} NC(V_s) ds\right] \qquad \text{(Equity)}$$

Investors choose a liquidation policy $\tau \in \mathcal{T}$ to maximize the value of equity, where $\mathcal{T}$ is the set of $(\mathcal{F}_t)$-stopping times.

Timing. For each interval of time $[t, t+\Delta)$, the timing is the following: first, shareholders decide whether to default and trigger the liquidation procedure; if default does not happen, bonds mature and new issuances are sold on the primary market (subperiod one); then, the secondary market opens and workers trade (subperiod two).

Bargaining Protocols. In the primary market, the firm makes a take-it-or-leave-it offer to bondless workers. In the secondary market, the bondless worker makes a take-it-or-leave-it offer to the bondholder.[4] Prices in the primary market, $\tilde{P}$, and secondary market, P, are allowed to be different depending on the surpluses of these transactions.

[4]This assumption is consistent with the restrictions imposed later on model parameters (i.e. $c/r \leq (1 - M)u + M\kappa$), which imply that the bondholder is always willing to trade at market prices. Knowing this, the bondless worker is able to extract the full surplus from every transaction.

3 Endogenous Trading Decisions

In a search economy, the bond price is determined by the extra utility that having a bond gives to workers, and therefore depends on workers' decision about whether to trade it or not. Those decisions are endogenous, and in turn, depend on the bond price. This mutual dependency is the source of a multiplicity of equilibria and prices: when workers believe that they will be able to use the bond in the future as a means of exchange for the consumption good, they are also willing to accept it immediately as a form of payment for their production activities; on the other hand, if the asset has no liquidity value, only its fundamentals–coupon rate and default risk–matter.

Denote by Π_1 the population probability that a worker holding the bond in a single-coincidence meeting is willing to exchange it for the consumption good. Similarly, denote with Π_0 the population probability that a worker is willing to accept the bond in exchange for producing one unit of the commodity. Taking those probabilities as given, workers maximize their lifetime utility by choosing whether to trade in the event of a single-coincidence meeting.

Proposition 1 (Value Functions). *The lifetime values $W_0(V_t)$ and $W_1(V_t)$ for the two workers, that are respectively not endowed and endowed with a bond, are the solutions of the following system of HJB equations:*

$$rW_0 = \sup_{\pi_0 \in [0,1]} \Big\{ \beta\delta(u-\kappa) + \beta\theta M \pi_0 \Pi_1 \big(P - \kappa\big) + \mu V W_0' + \frac{\sigma^2 V^2}{2} W_0'' \Big\}, \quad \text{(HJB0)}$$

$$rW_1 = \sup_{\pi_1 \in [0,1]} \Big\{ \beta\delta(u-\kappa) + c + \beta\theta(1-M)\pi_1 \Pi_0 \big(u - P\big) + \mu V W_1' + \frac{\sigma^2 V^2}{2} W_1'' \Big\}, \quad \text{(HJB1)}$$

where the bond price in the secondary market is given by $P(V_t) = W_1(V_t) - W_0(V_t)$, and derivatives are denoted with $W_i' := \frac{\partial W_i(V)}{\partial V}$ and $W_i'' := \frac{\partial^2 W_i(V)}{\partial V^2}$.

Value functions (HJB0-HJB1) can be visualized in their tree form in Figure 4 in the Appendix. The term $\beta\delta(u-\kappa)$ captures the expected utility of a double-coincidence meeting, which is the same for both agents. In addition to that, the bondholder receives the coupon payment c. These terms are unaffected by the liquidity value of the asset. The terms $\beta\theta(1-M)\Pi_0\big(u-P\big)$ and $\beta\theta M\Pi_1\big(P-\kappa\big)$ are the expected utility from participating to the trade: a bondholder gives up the capital gain for consuming immediately, whereas the agent without bond receives the value of the asset in exchange for paying the production cost. The remaining Ito terms describe how the values are expected to evolve as a function of the state.

A bondholder is willing to trade the asset whenever $\beta\theta(1-M)\Pi_0\big(u-P\big) \geq 0$, which occurs whenever $P \leq u$ or $\Pi_0 = 0$, that is when either the utility from consumption is large enough, or when the trade will not be accepted anyway from the counterparty. An offered trade is accepted whenever $\beta\theta M\Pi_1\big(P - \kappa\big) \geq 0$, that is whenever the production cost is not too large ($P \geq \kappa$), or when $\Pi_1 = 0$. Consequently, gains from trade are possible if and only if $u \geq P \geq \kappa$.

Let us now introduce the definition of equilibrium of the secondary market. I will focus on symmetric equilibria in pure strategies.

Definition 1 (Equilibrium of the secondary market)**.** *An equilibrium is a pair of subjective trading probabilities $\pi_i(V)$ for $i \in \{0,1\}$, a pair of trading probabilities in the population $\Pi_i(V)$ for $i \in \{0,1\}$, a pair of value functions $W_i(V)$ for $i \in \{0,1\}$, and a price function $P(V)$ such that:*

- *given the trading probabilities and the price function, the value functions are the solutions of (HJB0-HJB1) for almost every V;*
- *given the trading probabilities in the population and the price function, the subjective trading probabilities are admissible and maximize (HJB0-HJB1) for almost every V;*
- *subjective trading probabilities equal probabilities in the population:*

  $$\pi_i(V) = \Pi_i(V)$$

  for almost every V and $i \in \{0,1\}$;
- *the price function is consistent with value functions:*

  $$P(V) = W_1(V) - W_0(V)$$

  for almost every V.

The equilibrium imposes consistency between individual behavior (π_i) and beliefs about other agents' behavior (Π_i). These beliefs can in principle be complicated functions with arbitrary dependency on the state. We will show now that, in any equilibrium, trading probabilities must take a simple piecewise-constant form.

Lemma 1 (Subjective Trading Probabilities)**.** *Assume that $c/r \leq (1-M)u + M\kappa$. There exist two admissible controls π_i for $i \in \{0,1\}$, and a unique trading threshold $V_e \in [V_b, \infty]$ such that for almost every V*

$$\pi_1(V) = 1$$

and

$$\pi_0(V) = \begin{cases} 1 & \textit{if } \; V \geq V_e \\ 0 & \textit{if } \; V < V_e \end{cases}$$

maximize (HJB1) and (HJB0) respectively.

The subjective trading probabilities are the optimal controls for the system of equations (HJB0-HJB1), which, in pure strategies, take a simple bang-bang form.[5] As shown in the proof in Appendix A.3, the threshold is defined as $V_e := P^{-1}(\kappa)$ whenever $P(V)$ intersects κ for some $V < \infty$, and $V_e := \infty$ otherwise. Continuity and strict monotonicity of the price function ensure that the threshold is well-defined and unique. The assumption on the coupon rate $c/r \leq (1-M)u + M\kappa$ implies that a bondholder is always willing to trade the bond in exchange for the commodity.[6] This assumption will be maintained throughout the paper.

Whenever $V_e < \infty$, as will be clarified later, the trading threshold splits the state space into two regions of trading and not trading; when V is large, that

[5] Equilibria in mixed strategies exist but are less interesting for practical purposes because the price would be constant for a non-degenerate region of the state space. Also, a complete characterization would be involved as the roots of the characteristic equation for the bond price would not be piecewise constant.

[6] A weaker sufficient condition is $u \geq C(1)/R(1)$, with $R(1) = r + \beta\theta$ and $C(1) = c + \beta\theta\big((1-M)u + M\kappa\big)$, which is implied by the condition of the proposition. The tighter condition additionally ensures that $C(1)/R(1) \geq c/r$, which will be used later.

is when the firm is distant from default, the bond is traded, whereas when the firm is close to default, the asset is not accepted anymore in trades. This is the novel sense in which, in a search economy, fundamental risk affects liquidity: an exogenous deterioration of the fundamentals of the underlying asset, which brings the firm closer to default, makes the asset less appealing in trades; for a large enough decrease, the asset switches–discontinuously–from being traded into not being traded at all.

Let us now move to the characterization of the price. To ease the notation, I will denote with $\Pi(V) := \Pi_0(V) = \pi_0(V)$ the equilibrium (subjective and in the population) probability of accepting trades. Subtracting the dynamics of the value for the bondless agent from that of the bondholder leads to an asset pricing equation for the bond.

Proposition 2 (Bond Price)**.** *The bond price $P(V)$ satisfies:*

$$\begin{aligned} R(V)P(V) &= C(V) + \mu V P'(V) + \frac{\sigma^2 V^2}{2} P''(V), \qquad && V > V_b \\ P(V) &= \frac{(1-\alpha)}{r-\mu} V_b, && V \leq V_b \end{aligned} \tag{HJBP}$$

where $R(V) \equiv R(\Pi(V))$ and $C(V) \equiv C(\Pi(V))$ depend on the state only through the trading probabilities.

The bond price, given by the difference between the value for the bondholder and the value for the bondless worker, satisfies an asset pricing equation, where $R_t \equiv R(V_t)$ is the time-varying discount rate and $C_t \equiv C(V_t)$ is the time-varying dividend. Since C_t/R_t is bounded, it follows from the Feynman-Kac formula that the bond price is the expected discounted value of future dividends, as one would expect from standard asset pricing theory.

The time-varying components of interest rate and dividend show up because of the liquidity value of the asset. In particular, plugging in the trading decisions from lemma 1 and imposing equilibrium consistency, the discount rate becomes:

$$R(\Pi) = r + \beta\theta\Pi, \tag{Discount rate}$$

thus depends on the state V_t only through the trading decision Π. The additional discounting $\beta\theta(1-M)\Pi$ comes from the (negative) value for the bondholder that realizes when the bond is traded in single-coincidence meetings. The discounting $\beta\theta M\Pi$ comes from the (positive) value for the bondless worker when receiving the bond. Together, these two terms lower the price of the bond by increasing the effective discount rate. Interestingly, the additional spread $\beta\theta\Pi(V_t)$ is reminiscent of the liquidity premium from intensity-based models as in Amihud and Mendelson (1986), where $\beta\theta$ is the Poisson probability of the liquidity event. However, differently from that paper, the liquidity premium is here endogenous and becomes zero whenever the asset is not traded ($\Pi(V_t) = 0$).

Similarly for the dividend, plugging in the trading decisions, we get:

$$C(\Pi) = c + \beta\theta\Pi\big((1-M)u + M\kappa\big), \tag{Dividend}$$

which again depends on the state only through trading decisions. The terms in addition to the pure coupon rate capture the utility benefit (for the bondholder) and cost (for the bondless worker) from searching. Together, these additional terms raise the price of the bond by increasing the utility value of holding the asset.

4 Equilibria of the Secondary Market

This setting gives rise to a novel type of equilibrium–which will be referred to as a *safe-trade equilibrium*–in which the asset is traded if and only if it is safe, and switches endogenously from being traded to not being traded for a sufficiently drastic deterioration of fundamentals. The following definition formalizes the types of equilibria that will be relevant for the characterization of the search economy.

Definition 2 (Equilibrium Types)**.** *An equilibrium of the search economy is:*

- *a* ***no-trade equilibrium*** *if* $\Pi(V) = 0$ *for almost every* V*.*

- *a* ***safe-trade equilibrium*** *if there exists a trading threshold* $V_e > V_b$ *such that* $\Pi(V) = 0$ *for almost every* $V < V_e$ *and* $\Pi(V) = 1$ *for almost every* $V \geq V_e$.

The presence of beliefs regarding other workers' actions generates a coordination problem that leads to a multiplicity of equilibria and equilibrium prices. This is a well-understood feature of search economies as in Diamond (1984) and Diamond and Fudenberg (1989). To clarify the intuition for the multiplicity in this model, suppose that $V_e < \infty$ and take any $V > V_e$ large enough: at this distance to default, workers are willing to accept the bond in trades and $\pi(V) = \Pi(V) = 1$; for this to be possible, it must be that $P(V) \geq \kappa$, that is individual rationality is satisfied. For V large enough, the price approaches $C(\Pi)/R(\Pi)$, therefore $C(1)/R(1) > \kappa$ is sufficient to guarantee that there exists such a trading threshold $V_e < \infty$. However, suppose now that, at the same V, $\pi(V) = \Pi(V) = 0$, that is the bond is not accepted in trades; for individual rationality to be satisfied, it must be that $P(V) < \kappa$. Since V is large, the price approaches $C(\Pi)/R(\Pi)$, therefore $C(0)/R(0) = c/r < \kappa$ is sufficient to guarantee that $V < V_e$. As V was taken arbitrarily large, this implies $V_e = \infty$.

The above argument highlights the key point of the multiplicity: when the condition $C(1)/R(1) > \kappa > c/r$ is satisfied, multiple equilibrium prices are possible depending on workers' beliefs.

Proposition 3 (Characterization of Equilibria)**.** *Suppose that* $C(1)/R(1) > \kappa \geq c/r$. *For a given default threshold* V_b*, there exist two (and only two) symmetric equilibria in pure strategies. The first is a no-trade equilibrium with price function given by:*

$$P^{nt}(V; V_b) = \frac{c}{r}\left[1 - \left(\frac{V}{V_b}\right)^{-\gamma_2}\right] + \frac{(1-\alpha)}{r-\mu} V_b \left(\frac{V}{V_b}\right)^{-\gamma_2}, \qquad V \geq V_b.$$

The second is a safe-trade equilibrium with price function given by:

$$P^{st}(V;V_b) = \begin{cases} \left(\frac{c}{r}\left[1-\left(\frac{V}{V_b}\right)^{\gamma_1}\right] + \frac{(1-\alpha)}{r-\mu}V_b\left(\frac{V}{V_b}\right)^{\gamma_1}\right)\left[1-\frac{V^{-\gamma_2}V_b^{\gamma_1}-V^{\gamma_1}V_b^{-\gamma_2}}{V_e^{-\gamma_2}V_b^{\gamma_1}-V_b^{-\gamma_2}V_e^{\gamma_1}}\right] + \\ +\kappa\left(\frac{V^{-\gamma_2}V_b^{\gamma_1}-V^{\gamma_1}V_b^{-\gamma_2}}{V_e^{-\gamma_2}V_b^{\gamma_1}-V_b^{-\gamma_2}V_e^{\gamma_1}}\right) & \text{if } V_b \leq V < V_e \\ \\ \frac{C(1)}{R(1)}\left[1-\left(\frac{V}{V_e}\right)^{-\Gamma_2}\right] + \kappa\left(\frac{V}{V_e}\right)^{-\Gamma_2} & \text{if } V \geq V_e, \end{cases}$$

where $\gamma_1 > 0 > -\gamma_2$ are the positive and negative roots of the characteristic equation $\frac{\sigma^2}{2}\gamma^2 + (\mu - \sigma^2/2)\gamma - r = 0$ and $\Gamma_1 > 0 > -\Gamma_2$ are the positive and negative roots of the characteristic equation $\frac{\sigma^2}{2}\Gamma^2 + (\mu - \sigma^2/2)\Gamma - (r + \beta\theta) = 0$.

Finally, the trading threshold $V_e < \infty$ is uniquely pinned down by the smooth pasting condition:

$$\lim_{V \uparrow V_e} (P^{st})'(V) = \lim_{V \downarrow V_e} (P^{st})'(V).$$

The price in the no-trade equilibrium corresponds to that in Leland (1994), which is a linear combination between the payoff of a riskless coupon bond and the repayment at default, where the weight

$$\psi(V, V_b, \infty, r) = (V/V_b)^{-\gamma_2}$$

is the expected discounted value of an indicator function for the event of reaching V_b, given the initial state V.[7] This follows from the fact that, in a no-trade equilibrium, the asset has no liquidity value thus the price is only determined by coupon payments, repayment at default, and default risk.

The price in the safe-trade equilibrium generalizes the value of money as in Kiyotaki and Wright (1993) and the value of capital as in Lagos and Rocheteau (2008) to allow for fundamental risk. Even when not traded ($V < V_e$), the bond price depends on its liquidity through the probability that the asset will

[7]See Stokey (2008), proposition 5.3.

be traded in the future. Such probability is given by

$$\psi(V, V_e, V_b, r) = \frac{V^{-\gamma_2} V_b^{\gamma_1} - V^{\gamma_1} V_b^{-\gamma_2}}{V_e^{-\gamma_2} V_b^{\gamma_1} - V_b^{-\gamma_2} V_e^{\gamma_1}},$$

which is the expected discounted value of an indicator function for the event of reaching V_e before V_b is reached, given the initial state V. The smaller $\psi(V, V_e, V_b, r)$, the closer the bond price is to the value of an illiquid coupon bond. When the distance to default is large ($V > V_e$), the bond price oscillates between the value of a liquid riskless coupon bond, $C(1)/R(1)$, and the production cost κ. The weight

$$\psi(V, V_e, \infty, R(1)) = (V/V_e)^{-\Gamma_2}$$

is the probability–discounted at rate $R(1)$–of hitting the trading threshold, given the initial state.

The fact that the bondless worker is indifferent between trading and not trading at the trading threshold V_e leads to a smooth-pasting condition, which pins down uniquely the threshold.

The bond prices as functions of the fundamental in the two equilibria can be visualized in Figure 2: the extra liquidity value makes the price in the safe-trade equilibrium always larger than the price in the no-trade equilibrium. This result is formalized in the next corollary.

Corollary 1 (Liquidity Value)**.** *For a given default threshold V_b, it holds that:*

$$P^{st}(V; V_b) > P^{nt}(V; V_b),$$

for almost every $V > V_b$.

The severity of search frictions of the economy, parameterized by the probabilities of meeting ($\beta\theta$) and the fraction of bondholders in the economy (M), impacts the bond valuation through its liquidity value, given by $P^{st}(V) - P^{nt}(V)$. In particular, the next corollary shows that a weakening of the frictions always increases the liquidity value.

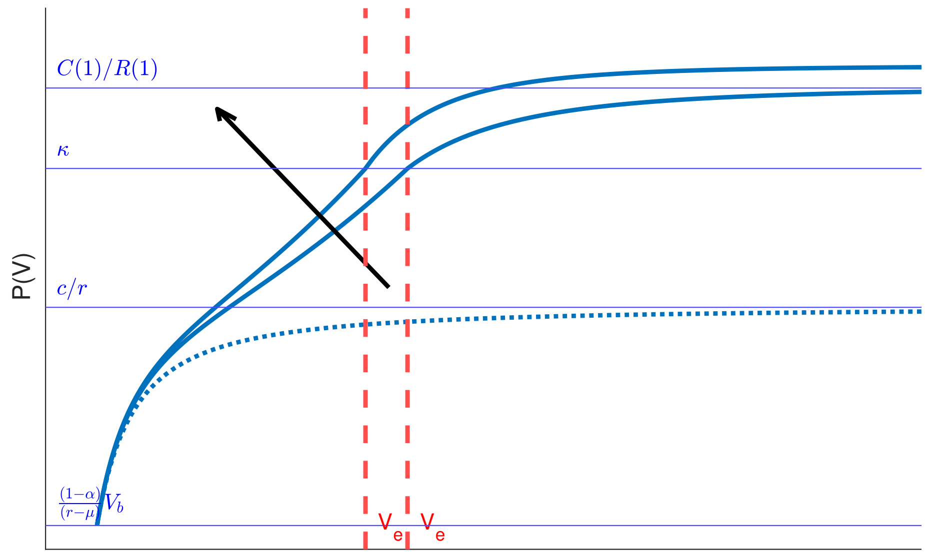


Figure 2: Price functions for a reduction in search frictions as in corollary 2. The blue full line is the price in the safe-trade equilibrium. The blue dashed line is the price in the no-trade equilibrium. The red dashed vertical line is the trading threshold. The black arrow illustrates the increase in price due to the reduction in search frictions.

Corollary 2 (Comparative Statics)**.** *For a given default threshold V_b, a reduction in the search frictions has no impact on the price in the no-trade equilibrium, but increases the price in the safe-trade equilibrium:*

$$\frac{\partial P^{st}}{\partial \beta}\bigg|_{V_b} > 0; \ \frac{\partial P^{st}}{\partial \theta}\bigg|_{V_b} > 0; \ \frac{\partial P^{st}}{\partial M}\bigg|_{V_b} < 0, \qquad V > V_b$$

and reduces the trading threshold:

$$\frac{\partial V_e}{\partial \beta}\bigg|_{V_b} < 0; \ \frac{\partial V_e}{\partial \theta}\bigg|_{V_b} < 0; \ \frac{\partial V_e}{\partial M}\bigg|_{V_b} > 0.$$

The reduction in search frictions is plotted in Figure 2: for every value of the fundamental, the price shifts to the left, implying a reduction in the trading threshold. A lower threshold, in turn, increases $\psi(V, V_e, V_b, r)$ for every $V < V_e$, making it more likely for the bond to be traded in the future.

Finally, proposition 3 is derived under the assumption that the production cost, κ, is larger than the value of a riskless illiquid bond, c/r. When the coupon rate is large enough, agents are always willing to accept the bond in trades when the asset is safe enough, leading to a unique equilibrium.

Proposition 4 (Price Uniqueness)**.** *Suppose that $c/r > \kappa$. Then the unique symmetric equilibrium in pure strategies is a safe-trade equilibrium.*

The proposition shows that it is not always the case that multiple equilibria arise in this model. In fact, when departing from the case $C(1)/R(1) > \kappa > c/r$, uniqueness is generally achieved. In particular, following the same steps as the proof for proposition 4, one could show that $C(1)/R(1) < \kappa$ implies that trades can never be supported and the unique equilibrium is of the no-trade type.

5 Equilibria of the Primary Market

At the beginning of subperiod one, bonds expire at rate $1/T$, and bondholders receive the principal; the M/T expired bonds are replaced immediately and purchased at market price by the $(1-M)$ share of workers that are not holding asset inventories. Denote with $\tilde{W}_0$ the value for the bondless worker in subperiod one, and with $\tilde{W}_1$ the value for a bondholder in subperiod one. The values for the two workers are given by:

$$\tilde{W}_0(V_t) = e^{-r\Delta}\,\mathbb{E}_t\left\{\frac{M}{T(1-M)}[\tilde{W}_1(V_{t+\Delta}) - \tilde{P}(V_{t+\Delta})] + \left(1-\frac{M}{T(1-M)}\right)W_0(V_{t+\Delta})\right\}$$
$$\tilde{W}_1(V_t) = e^{-r\Delta}\,\mathbb{E}_t\left\{\frac{1}{T}(\tilde{W}_0(V_{t+\Delta}) + p) + \left(1-\frac{1}{T}\right)W_1(V_{t+\Delta})\right\},$$

where the bond price in the primary market is given by $\tilde{P}(V_t) = \tilde{W}_1(V_t) - \tilde{W}_0(V_t)$. Letting $\Delta \to 0$ implies $\tilde{W}_0 = W_0$ and $\tilde{W}_1 = W_1 + \frac{1}{T-1}(p-\tilde{P})$. The values for the two workers at the beginning of subperiod one reflect the fact that the bondless worker is indifferent between purchasing or not the bond from the issuing firm, whereas the bondholder carries rollover losses or gains whenever the bond expires above or below par. Substituting into the price:

$$\tilde{P} = \left(1 - \frac{1}{T}\right) P + \frac{1}{T} p, \tag{C}$$

thus the price in the primary market is a convex combination of the price in the secondary market and the repayment at maturity, where the weights reflect the probability of stochastic maturity.

Let us now discuss the definition of equilibrium for the primary market: investors take the price of the bond as given and choose a liquidation policy to maximize the equity valuation.

Definition 3 (Equilibrium of the primary market)**.** *An equilibrium is an equity value $E(V)$, a stopping time $\tau \in \mathcal{T}$, and a price function for the primary market $\tilde{P}(V)$, such that:*

- *given the price function, the stopping time is admissible and maximizes the equity value for almost every V;*
- *the price function is consistent with the price in the secondary market, that is condition (C) holds for almost every V.*

As the cash flow increases in the value of the fundamental but depends negatively on the fixed coupon rate, for a sufficiently weak fundamental, the cash flow becomes negative. Shareholders may nevertheless prefer to continue operations, for the prospect that a future recovery will make the cash flow positive again. At some sufficiently low value of the fundamental, the prospect of such recovery is dim enough to warrant immediate liquidation. Moreover, if liquidation is optimal at a particular value of the fundamental, it is also optimal at any lower value. This implies that the optimal liquidation time is the first time that the fundamental value falls below the default threshold.

Proposition 5 (Equity Value)**.** *The equity value $E(V)$ satisfies:*

$$\begin{aligned} rE(V) &= V - \zeta c + \frac{M}{T}\Big(\tilde{P}(V) - p\Big) + \mu V E'(V) + \frac{\sigma^2}{2}V^2 E''(V), \quad V > V_b \\ E(V) &= 0, \qquad V \leq V_b \end{aligned} \tag{HJBE}$$

with smooth pasting condition $E'(V_b) = 0$.

Proposition 5, which describes the evolution of the equity value, shows the dependence of the equity value on rollover gains/losses, $\frac{M}{T}\Big(\tilde{P}(V) - p\Big)$. When the bond price is larger than the principal ($\tilde{P} > p$), the firm enjoys a positive cash flow from rolling over the debt.

The equilibrium imposes consistency between the price in the primary market and that in the secondary market. As investors take the price in the secondary market as given, consistency implies that the multiplicity of prices due to the search friction generates a multiplicity of equity valuations: for the same fundamental, a firm issuing liquid bonds will have a higher valuation than a firm issuing illiquid bonds. Denote with E^{nt}, V_b^{nt} and with E^{st}, V_b^{st} the equity values and default thresholds when the economy is in the no-trade equilibrium and safe-trade equilibrium, respectively.

Proposition 6 (Characterization of Equilibria)**.** *The equity value in the no-trade equilibrium is given by:*

$$\begin{aligned} E^{nt}(V) =& \frac{V}{r-\mu} - \frac{V_b^{nt}}{r-\mu}\left(\frac{V}{V_b^{nt}}\right)^{-\gamma_2} - \zeta\frac{c}{r}\left[1 - \left(\frac{V}{V_b^{nt}}\right)^{-\gamma_2}\right] \\ &+ \frac{M}{T}\left(1 - \frac{1}{T}\right)\frac{1}{r}\left(\frac{c}{r} - p\right)\left[1 - \left(\frac{V}{V_b^{nt}}\right)^{-\gamma_2}\right] \\ &+ \frac{M}{T}\left(1 - \frac{1}{T}\right)\left(\frac{c}{r} - \frac{(1-\alpha)}{r-\mu}V_b^{nt}\right)\left(\frac{V}{V_b^{nt}}\right)^{-\gamma_2}\ln\left(\frac{V}{V_b^{nt}}\right)^{\frac{1}{\mu-\sigma^2/2}}. \end{aligned}$$

where $-\gamma_2$ is the negative root of the characteristic equation $\frac{\sigma^2}{2}\gamma^2 + (\mu - \sigma^2/2)\gamma -$

$r = 0$ and the default threshold is given by:

$$V_b^{nt} = \frac{\gamma_2 \zeta \frac{c}{r} + \gamma_2 \frac{M}{T}\left(1 - \frac{1}{T}\right)\frac{1}{r}\left(p - \frac{c}{r}\right) - \frac{M}{T}\left(1 - \frac{1}{T}\right)\frac{1}{\mu - \sigma^2/2}}{\frac{1+\gamma_2}{r-\mu} - \frac{(1-\alpha)}{r-\mu}\frac{M}{T}\left(1 - \frac{1}{T}\right)\frac{1}{\mu - \sigma^2/2}}.$$

For almost all $V > V_b^{st}$, the equity value in the safe-trade equilibrium (equation 1 in Appendix A.7) is larger:

$$E^{st}(V) > E^{nt}(V),$$

and the default threshold is smaller:

$$V_b^{st} < V_b^{nt}.$$

The equity valuation in the no-trade equilibrium coincides with Duffie and Lando (2001) whenever there is no rollover risk, which is nested as a special case when either debt has infinite maturity ($T \to \infty$), or there is a vanishing number of bonds in the economy ($M \to 0$). Similarly to He and Xiong (2012c) and He and Milbradt (2014), debt rollover increases the probability of default whenever rollover losses are large on average, which happens for example when the principal payment is larger than the discounted coupon payment, i.e. $p > c/r$. Moreover, the liquidity value of the bond increases the equity valuation and therefore reduces the probability of default. This mechanism creates the possibility of a *default-liquidity loop*, in which an exogenous change in search frictions not only affects the price in the secondary market but also the equity valuation and default decisions through the bond price of the primary market.[8] The next corollary illustrates the mechanism.

Corollary 3 (Default-Liquidity Loop)**.** *A reduction in the search frictions increases the equity in the safe-trade equilibrium, that is:*

$$\frac{\partial E^{st}}{\partial \beta} > 0; \ \frac{\partial E^{st}}{\partial \theta} > 0;$$

[8] The same mechanism is present in the different classes of models studied in He and Xiong (2012c) and He and Milbradt (2014).

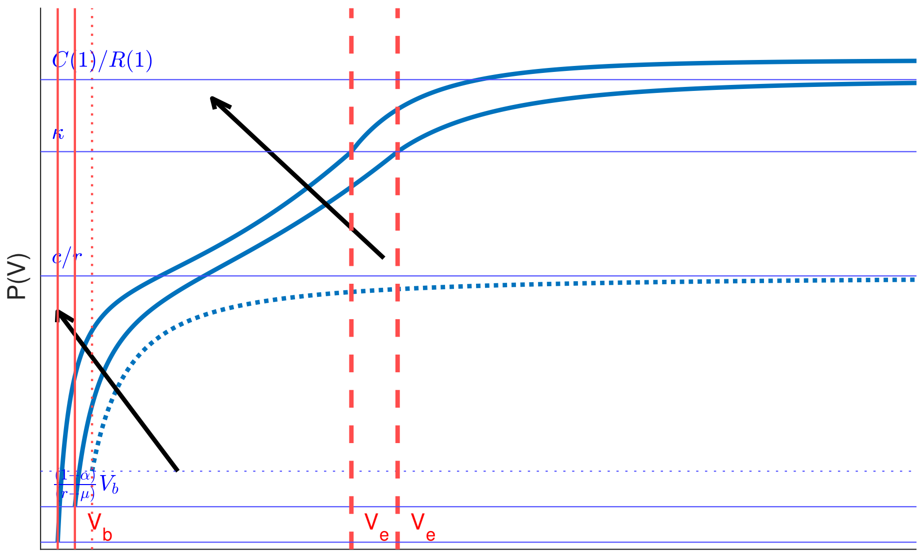


Figure 3: Price functions for a reduction in search frictions as in corollary 3. The blue full line is the price in the safe-trade equilibrium. The blue dashed line is the price in the no-trade equilibrium. The red dashed vertical line is the trading threshold. The red full vertical line is the default threshold. The black arrow illustrates the increase in price due to the reduction in search frictions and the endogenous decrease in the default threshold.

for almost every $V > V_b^{st}$, and reduces the default threshold, that is:

$$\frac{\partial V_b^{st}}{\partial \beta} < 0;\ \frac{\partial V_b^{st}}{\partial \theta} < 0.$$

As shown in Figure 3, a reduction in the search frictions affects directly the bond price (as in corollary 2) and indirectly through the change in the default threshold. When the latter further increases the bond price, that is when the price increases for a reduction in the default threshold, a feedback loop between liquidity and default takes place:[9]

$$\frac{\partial P^{st}}{\partial \beta} = \underbrace{\frac{\partial P^{st}}{\partial \beta}\Big|_{V_b}}_{>0} + \underbrace{\frac{\partial P^{st}}{\partial V_b^{st}}}_{<0}\underbrace{\frac{\partial V_b^{st}}{\partial \beta}}_{<0}.$$

The same mechanism leads to a uniform additional increase in the trading probability and a further reduction in the trading threshold:

$$\frac{\partial V_e}{\partial \beta} = \underbrace{\frac{\partial V_e}{\partial \beta}\Big|_{V_b}}_{<0} + \underbrace{\frac{\partial V_e}{\partial V_b^{st}}}_{>0}\underbrace{\frac{\partial V_b^{st}}{\partial \beta}}_{<0}.$$

6 Discussion

Let us now discuss some stylized extensions of the model to show the robustness and applicability of the main results.

Sovereign Debt. The model can be readily used to rationalize features of the US treasury bond market. By relabelling the fundamental as the aggregate GDP and the issuing firm as the government, the model generates the positive correlation between liquidity premia (or negative correlation of the AAA-Treasury spread) and the debt-to-GDP ratio for the US government, which has been documented in Krishnamurthy and Vissing-Jorgensen (2012). Model's predictions are also in line with Chaumont (2020), which studies the interaction between liquidity and fundamental risk for the treasury bond market and argues that illiquidity increases with default risk and accounts for 10-50 percent of the credit spread.

Flights to safety. Let us now add a safe competing asset to discuss flight-to-safety/liquidity behaviors. To ease the notation, assume that the asset does

[9] A sufficient condition for $\frac{\partial P}{\partial V_b} < 0$ is that $\alpha \approx 1$, which implies that recovery value does not decrease faster than the default boundary. Generally, given the parameters, one can show that the price decreases in the default boundary if and only if $V_b < \bar{V}_b$, where $\bar{V}_b$ is a positive number.

not carry a coupon and is riskless (fiat money). Money is a competing medium of exchange with the risky bond in the sense that in any period only one of the two (with some probability) enjoys the "privilege" of being the medium of exchange:

$$\Pi^s(V) + \Pi^r(V) = 1, \quad \text{a.e. } V$$

where $\Pi^j \geq 0$ is the probability that asset j has the privilege. Consider the case in which both assets are initially accepted in trades, that is $\Pi^j(V_{t-}) > 0$ for $j \in \{s, r\}$ at V_{t-} and assume that Π^r is a monotonically increasing function as before. Denote with $0 < m^s \leq M$ the quantity of the safe asset.

Corollary 4 (Flight to safety)**.** *A deterioration in the fundamental V_t leads to an increase in the trading probability of the safe asset. Moreover, the price of the safe asset is given by:*

$$P^s(V_t) = \frac{\beta\theta(1 - \Pi^r(V_t))\left[(1 - m^s)u + m^s\kappa\right]}{r + \beta\theta(1 - \Pi^r(V_t))}, \qquad \text{(Safe Asset)}$$

which decreases in V_t.

The corollary shows that, when the fundamental of the risky asset decreases, agents endogenously switch to the competing media of exchange, which is the safe asset. This implies that the price of the safe asset increases as well because its liquidity value increases whenever the competing medium becomes less liquid. Equivalently, fiat money displays countercyclical price fluctuations and risk premia are procyclical. Finally, interpreting the safe asset as a US treasury zero-coupon bond and the risky asset as a corporate bond, the corollary generates flights to safety as those discussed in Longstaff (2002).

Balance-sheet constraints. In applications, the inventory constraint that limits the ability of workers to hold more than one unit of the asset may appear unrealistic. In particular, when agents can adjust at the margin the quantity $q^B(V) \geq 0$ of bonds that they offer in trades, assets would always be traded for any fundamental risk as long as:

$$P(V) \cdot q^B(V) \geq \kappa.$$

As the price declines, the quantity of bonds offered can be increased to make it worth it for the bondless worker to still accept the trade. However, as long as workers are subject to a balance-sheet constraint that limits the number of bonds that they can hold ($q^B(V) \leq \bar{q}$), there still exists a trading threshold such that trades occur if and only if fundamentals are above the threshold:

$$V_e := P^{-1}(\kappa/\bar{q}).$$

For $\bar{q}$ large enough, $V_e < V_b$, and the only equilibrium that survives always involves trading. However, when the constraint is tight, results carry through unchanged.

Transaction costs. When the bondless worker is allowed to exchange only a fraction of the commodity ($0 \leq q^C(V) \leq 1$), trades always occur for any fundamental risk. As the price declines, the quantity of commodities can be decreased to make it worth it for the bondless worker to still accept the trade. In this case, the safe-trade equilibrium is obtained again by introducing a fixed cost for the transaction ($\bar{\kappa}$). In this case, trading occurs whenever:

$$P(V) \geq \kappa \cdot q^C(V) + \bar{\kappa},$$

and the new trading threshold becomes:

$$V_e := P^{-1}(\bar{\kappa}).$$

For $V > V_e$ small enough, a vanishing quantity of commodities is exchanged, and for $V < V_e$ no trades occur.

Unique Equilibrium of the Search Economy. In Section 5, the firm takes as given workers' beliefs and therefore the price in the primary market which has to satisfy the consistency condition (C). This leads to the possibility of inefficient equilibria with no trades, in which both workers and the firm are worse off. However, if the firm can post a "recommended" price for the newly issued debt, workers would use such public announcement to coordinate around the better equilibrium. Let us now introduce formally this equilibrium refinement.

Definition 4 (Equilibrium of the search economy)**.** *An equilibrium is a posted price function $\hat{P}(V)$, and price functions for the primary $\tilde{P}(V)$ and secondary $P(V)$ markets such that:*

- *given $P(V)$, the secondary market is in equilibrium;*
- *given $\tilde{P}(V)$, the primary market is in equilibrium and the consistency condition (C) holds for almost every V;*
- *the posted price satisfies $\hat{P}(V) = \tilde{P}(V)$ for almost every V.*

The next corollary formalizes the uniqueness result for the search economy.

Corollary 5 (Uniqueness of Equilibrium)**.** *In the search economy with posted price, the unique equilibrium is a safe-trade equilibrium.*

When the firm can affect workers' beliefs by sending a public message in the form of a posted price, the efficient equilibrium with trading can be implemented uniquely. Broadly speaking, the corollary can be interpreted as showing that the multiplicity of equilibria in the secondary markets is generally less likely to be present in models in which the issuing firm is large with respect to the size of the market.

7 Conclusions

This paper develops an asset pricing framework with search frictions in the secondary market. I show that, when trading decisions are endogenous, the asset is accepted in decentralized trades if and only if it is safe, meaning that the issuing firm has a low probability of defaulting on its liabilities. This gives rise to a novel type of monetary equilibrium characterized by a trading threshold for the profitability of the underlying asset. When profitability is above the threshold, the asset is accepted in trades, whereas the market freezes as soon as the profitability falls below it.

This channel rationalizes empirical regularities documented by the literature, such as the negative correlation between liquidity premia and leverage ratios and business-cycle fluctuations in liquidity premia, as well as market freezes and flight-to-safety/liquidity behavior in the occurrence of large deteriorations of aggregate fundamentals.

When the issuing firm rolls over debt, the liquidity value of the asset in the secondary market feeds back into the defaulting decision. In particular, if the asset loses the privilege, the equity valuation decreases and default occurs more frequently. By the same mechanism, an exogenous worsening of the search frictions, which leads to a decrease in the liquidity value of the asset, generates rollover losses and higher default probabilities. In turn, more frequent defaults further reduce the asset value leading to a negative price spiral.

I discuss the generality and applicability of the main results of the paper, which would hold true in a large class of applications and extensions. In particular, when extended to include a competing safe asset (e.g. fiat money), the model predicts procyclical risk premia and flight-to-safety behavior. Finally, the model is tractable and leads to closed-form expressions that can be easily incorporated into a wide range of applications.

Appendix

A Proofs for the Main Text

A.1 Proof of proposition 1

Consider a small interval of time $\Delta > 0$. The value between t and $t+\Delta$ for the worker that does not hold the bond can be written recursively as:

$$\begin{aligned}W_0(V_t) = \sup_{\pi_0\in[0,1]} e^{-r\Delta}\,\mathbb{E}_t\Big\{&(1-\beta\Delta)W_0(V_{t+\Delta}) + \beta\Delta\delta(u-\kappa+W_0(V_{t+\Delta}))\\&+\beta\Delta\theta W_0(V_{t+\Delta}) + \beta\Delta\theta(1-M)W_0(V_{t+\Delta})\\&+\beta\Delta\theta M\Big[\Pi_1\pi_0(c\Delta+W_1(V_{t+\Delta})-\kappa)+(1-\Pi_1\pi_0)W_0(V_{t+\Delta})\Big]\\&+\beta\Delta(1-2\theta-\delta)W_0(V_{t+\Delta})+o(\Delta)\Big\},\end{aligned}$$

where the expectation is taken with respect to the filtration $\{\mathcal{F}_t\}$ generated by the Brownian motion $\{Z_t\}$. Hence for Δ sufficiently small:

$$\begin{aligned}r\Delta W_0(V_t) = \sup_{\pi_0\in[0,1]}\Big\{&\beta\Delta\delta(u-\kappa)+\beta\Delta\theta M\Pi_1\pi_0\big(c\Delta+\mathbb{E}_t\,W_1(V_{t+\Delta})\\&-\mathbb{E}_t\,W_0(V_{t+\Delta})-\kappa\big)+\big(\mathbb{E}_t\,W_0(V_{t+\Delta})-W_0(V_t)\big)+o(\Delta)\Big\}.\end{aligned}$$

Now divide the above by Δ, let $\Delta\to 0$, and apply Ito's lemma to get:

$$rW_0 = \sup_{\pi_0\in[0,1]}\Big\{\beta\delta(u-\kappa)+\beta\theta M\pi_0\Pi_1\big(W_1-W_0-\kappa\big)+\mu VW_0'+\frac{\sigma^2V^2}{2}W_0''\Big\}.$$

Similarly for the worker endowed with a bond:

$$\begin{aligned}W_1(V_t) = e^{-r\Delta}\,\mathbb{E}_t\Big\{&(1-\beta\Delta)(c\Delta+W_1(V_{t+\Delta}))+\beta\Delta\delta(c\Delta+u-\kappa+W_1(V_{t+\Delta}))\\&+\beta\Delta\theta(c\Delta+W_1(V_{t+\Delta}))+\beta\Delta\theta M(c\Delta+W_1(V_{t+\Delta}))\\&+\beta\Delta\theta(1-M)\Big[\pi_1\Pi_0(u+W_0(V_{t+\Delta}))+(1-\pi_1\Pi_0)(c\Delta+W_1(V_{t+\Delta}))\Big]\\&+\beta\Delta(1-2\theta-\delta)(c\Delta+W_1(V_{t+\Delta}))+o(\Delta)\Big\}.\end{aligned}$$

Hence for Δ sufficiently small:

$$r\Delta W_1(V_t) = \sup_{\pi_1 \in [0,1]} \Big\{ \beta\Delta\delta(u-\kappa) + c\Delta + \beta\Delta\theta(1-M)\pi_1\Pi_0\big(\mathbb{E}_t W_0(V_{t+\Delta}) - c\Delta - \mathbb{E}_t W_1(V_{t+\Delta}) + u\big) + \big(\mathbb{E}_t W_1(V_{t+\Delta}) - W_1(V_t)\big) + o(\Delta)\Big\}.$$

Now divide the above by Δ, let $\Delta \to 0$, and apply Ito's lemma to get:

$$rW_1 = \sup_{\pi_1 \in [0,1]} \Big\{ \beta\delta(u-\kappa) + c + \beta\theta(1-M)\pi_1\Pi_0\big(u - W_1 + W_0\big) + \mu V W_1' + \frac{\sigma^2 V^2}{2} W_1'' \Big\}.$$

The bond price is given by $P = W_1 - W_0$; this is implied by the bargaining protocol, with all the surplus from the transaction being acquired by the bondless worker. Finally, the verification step follows from standard arguments and the fact that payoffs are bounded.

A.2 Proof of proposition 2

Subtracting rW_0 from rW_1 and rearranging, we obtain:

$$(r + \beta\theta(1-M)\Pi_0 + \beta\theta M\Pi_1\pi_0)P = c + \beta\theta(1-M)u\Pi_0 + \beta\theta M\kappa\Pi_1\pi_0 + \mu V P' + \frac{\sigma^2 V^2}{2} P''$$

The dependence of $R(V) := (r + \beta\theta(1-M)\Pi_0(V) + \beta\theta M\Pi_1(V)\pi_0(V))$ and $C(V) := c + \beta\theta(1-M)u\Pi_0(V) + \beta\theta M\kappa\Pi_1(V)\pi_0(V)$ on the state depends only through the trading probabilities. Together with lemma 1 and equilibrium consistency $\pi_i = \Pi_i$, $i \in \{0, 1\}$ we get the result.

A.3 Proof of lemma 1, proposition 3, and corollary 1

The proof is derived in the following steps: first, guess that lemma 1 holds; second, solve in closed form for the equilibrium price for the two different equilibria with $V_e = \infty$ (no-trade) and $V_e < \infty$ (safe-trade); third, argue that smooth pasting must hold at the trading threshold; fourth, impose smooth pasting to derive the trading threshold and prove existence and uniqueness; fifth, verify that the guess is satisfied and the system (HJB0-HJB1) is maximized by the admissible controls at the equilibrium price; finally, conclude the proofs.

Step 1: Assuming that lemma 1 holds implies that $C(\Pi(V))$ and $R(\Pi(V))$ are piecewise constant for almost every V. In the no-trade equilibrium,

$V_e = \infty$, which implies that the coefficients are a.e. constant.

Step 2: By taking the change of variable $y := \ln(V)$, Equation (HJBP) becomes:

$$R(\Pi(y))P(y) = C(\Pi(y)) + mP'(y) + \frac{\sigma^2}{2}P''(y),$$

where $m := \mu - \frac{\sigma^2}{2}$. By step 1, coefficients are piecewise constant, implying that the homogeneous solutions of the ODE are $S_1(y) = e^{\Gamma_1 y}$ and $S_2(y) = e^{-\Gamma_2 y}$ for $y \in (\ln(V_e), \infty)$, where Γ_i $i = 1, 2$ are the solutions (in absolute values) of

$$\frac{\sigma^2}{2}\Gamma^2 + m\Gamma - (r + \beta\theta) = 0,$$

and $s_1(y) = e^{\gamma_1 y}$ and $s_2(y) = e^{-\gamma_2 y}$ for $y \in [\ln(V_b), \ln(V_e))$, where γ_i $i = 1, 2$ are the solutions (in absolute values) of

$$\frac{\sigma^2}{2}\gamma^2 + m\gamma - r = 0.$$

Since $\mu < r < R$, the equations lead to $\gamma_1 > 1 > 0 > -\gamma_2$ for $[\ln V_b, \ln V_e]$ and $\Gamma_1 > 1 > 0 > -\Gamma_2$ for $[\ln V_e, \infty)$. Hence the general solution takes the form:

$$P(V) = \begin{cases} a_0 + a_1 V^{\gamma_1} + a_2 V^{-\gamma_2} & \text{if } V \in [V_b, V_e) \\ A_0 + A_1 V^{\Gamma_1} + A_2 V^{-\Gamma_2} & \text{if } V \in (V_e, \infty), \end{cases}$$

where the constants have to be determined using boundary conditions.

Let us now distinguish in the two cases: no-trade equilibrium and safe-trade equilibrium. Standard boundary conditions for the price in the no-trade equilibrium are $P^{nt}(V_b) = \frac{(1-\alpha)}{r-\mu}V_b$ and $\lim_{V\to\infty} P^{nt}(V) = c/r$. This implies $a_0 = c/r$, $a_1 = 0$, and $a_2 = \left[\frac{(1-\alpha)}{r-\mu}V_b - \frac{c}{r}\right]V_b^{\gamma_2}$. Therefore we get the equation in proposition 3.

For the safe-trade equilibrium, boundary conditions are $P^{st}(V_b) = \frac{(1-\alpha)}{r-\mu}V_b$, $\lim_{V\to\infty} P^{st}(V) = \frac{C(1)}{R(1)}$, $\lim_{V\to V_e^-} P^{st}(V) = \lim_{V\to V_e^+} P^{st}(V)$, and $\lim_{V\to V_e^-} P^{st}(V) = \kappa$.

This implies $a_0 = \frac{c}{r}$, $A_0 = \frac{C(1)}{R(1)}$, $A_1 = 0$, and

$$\begin{cases} P^{st}(V_b) = \frac{c}{r} + a_1 V_b^{\gamma_1} + a_2 V_b^{-\gamma_2} = \frac{(1-\alpha)}{r-\mu} V_b \\ P^{st}(V_e) = \frac{c}{r} + a_1 V_e^{\gamma_1} + a_2 V_e^{-\gamma_2} = \kappa \end{cases}$$

$$\Rightarrow a_2 = \frac{V_e^{\gamma_2} \left[\kappa - \frac{c}{r}\left(1 - \left(\frac{V_e}{V_b}\right)^{\gamma_1}\right) - \frac{(1-\alpha)}{r-\mu} V_b \left(\frac{V_e}{V_b}\right)^{\gamma_1}\right]}{1 - \left(\frac{V_e}{V_b}\right)^{\gamma_1}\left(\frac{V_e}{V_b}\right)^{\gamma_2}}$$

$$\Rightarrow a_1 = \frac{(1-\alpha)}{r-\mu} V_b^{1-\gamma_1} - \frac{c}{r} V_b^{-\gamma_1} - a_2 V_b^{-(\gamma_1+\gamma_2)}$$

$$\Rightarrow A_2 = \left[\kappa - \frac{C(1)}{R(1)}\right] V_e^{\Gamma_2}.$$

Rearranging gives the equation in proposition 3.

Step 3: Suppose the threshold V_e has been chosen and the state is $V = V_e$. It follows from lemma 1 that, at the threshold, the bondless worker is indifferent between the two following strategies: (i) trade immediately and (ii) not trade immediately, wait for a short interval of time h, and then re-optimize. Re-optimization entails trading if the price increases and not trading if the price decreases. The return from the first strategy is simply $\Pi_1 = W_0(V_e)$. The return from the latter can be calculated using the random walk approximation, with $\Delta V = \pm\sigma\sqrt{h}$:

$$\Pi_2 = (1-p)W_0(V_e - \sigma\sqrt{h}) + pW_0(V_e + \sigma\sqrt{h}),$$

where $p := \frac{1}{2}\left[1 + \frac{\mu\sqrt{h}}{\sigma}\right]$ is the probability of an upward jump. Using the Taylor series expansion, we get:

$$\Pi_2 \approx W_0(V_e) - (1-p)W_0'(V_e^-)\sigma\sqrt{h} + pW_0'(V_e^+)\sigma\sqrt{h}$$

Indifference between the two strategies implies:

$$\Pi_1 - \Pi_2 = 0$$
$$\Leftrightarrow (1-p)W_0'(V_e^-) = pW_0'(V_e^+)$$

Letting $h \to 0$ and noticing that $\lim_{h\to 0} p = \frac{1}{2}$, implies $W_0'(V_e^-) = W_0'(V_e^+)$. Finally, using HJB0, the condition is satisfied only if $P'(V_e^-) = P'(V_e^+)$.

Step 4: The trading threshold for the case $V_e < \infty$ is obtained by imposing the smooth-pasting condition $P'(V_e^-) = P'(V_e^+)$. This implies that V_e must satisfy:

$$1 + \frac{a_1\gamma_1}{A_2\Gamma_2} V_e^{\Gamma_2+\gamma_1} = \frac{a_2\gamma_2}{A_2\Gamma_2} V_e^{\Gamma_2-\gamma_2}$$

Let us now argue existence and uniqueness of V_e. Existence follows from continuity of the price function and the fact that $\frac{(1-\alpha)}{r-\mu} V_b \leq \kappa < \frac{C(1)}{R(1)}$. For uniqueness, notice first that $\frac{C(1)}{R(1)} > \kappa$ and $P^{st}(V_e) = \kappa$ imply that $A_2 < 0$, hence P^{st} is strictly increasing for $V \in (V_e, \infty)$. Also, direct calculations show that $a_2 < 0$ because the numerator is positive and the denominator is negative. Since $a_0 = \frac{c}{r} < \kappa$ and $a_2 < 0$, then $a_1 > 0$, because otherwise it would contradict that $P^{st}(V_e) = \kappa$. Hence, $P'^{st}(V) = \gamma_1 a_1 V^{\gamma_1 - 1} - \gamma_2 a_2 V^{-(\gamma_2+1)} > 0$ and P^{st} is strictly increasing for $V \in [V_b, V_e)$. Since P^{st} is strictly increasing almost everywhere and it is continuous, smooth pasting ensures that V_e is unique.

Step 5: To verify the guess from lemma 1, it is enough to notice that the objective functions are linear in the controls, therefore the controls must be of the form

$$\pi_0(V) = \begin{cases} 1 & P(V) > \kappa \\ \in [0,1] & P(V) = \kappa \\ 0 & P(V) < \kappa \end{cases} \quad \text{and} \quad \pi_1(V) = \begin{cases} 1 & P(V) < u \\ \in [0,1] & P(V) = u \\ 0 & P(V) > u \end{cases}$$

For π_1, the assumption on parameters ensure that $P(V) < u$ for almost all V. For π_0, strict monotonicity of the price function implies

$$\pi_0(V) = \begin{cases} 1 & V > P^{-1}(\kappa) =: V_e \\ 0 & V < P^{-1}(\kappa) =: V_e \end{cases}$$

for almost all V. Since controls are piecewise constant, they are admissible. Moreover, since the maximum is attained, standard verification theorems imply that the value function coincides with the lifetime value for the two agents.

For proposition 3, we are left to prove that only the two equilibria above exist (in pure strategies). This follows from the uniqueness of a strong solution to equation (HJBP) (see e.g. Theorem 5.2.1 in Øksendal (2003)), and linearity of the optimal control problem for the system (HJB0-HJB1), which implies that $\pi_0(V)$ can either only be constant at zero or take a bang-bang form (no other cases are possible because of the choice of the boundary conditions). Finally, the claim $P^{st}(V) > P^{nt}(V)$ for almost every $V \in (V_b, \infty)$ can be verified by direct calculations, or by using the Feynman-Kac formula and the boundary conditions $P^{st}(V_b) = P^{nt}(V_b)$ and $\lim_{V\to\infty} P^{st}(V) = \frac{C(1)}{R(1)} > \frac{c}{r} = \lim_{V\to\infty} P^{nt}(V)$. This concludes the proof for proposition 3 and corollary 1.

A.4 Proof of corollary 2

First, notice that β and θ increase both $C(1)$ and $R(1)$, whereas M decreases $C(1)$ and does not enter $R(1)$. As long as $C(1)/R(1)$ increases, the result follows from noticing that the solution of equation (HJBP) is increasing in $C(1)/R(1)$.

To show this, suppose that it is not true to derive a contradiction. Consider two different solutions $\bar{P}$ and $\underline{P}$ for $\bar{C}/\bar{R} > \underline{C}/\underline{R}$. First, since $\bar{R} > \underline{R}$, from the characteristic equation of (HJBP), it follows that the negative root must be smaller (larger in absolute value) for $\bar{P}$ than $\underline{P}$. This implies that, for $V \to \infty$, $0 > \bar{P}'' > \underline{P}''$, because $\underline{P}(V) \approx \frac{\underline{C}}{\underline{R}} + \left(\frac{V}{const}\right)^{-\underline{\Gamma}_2}$ and $\bar{P}(V) \approx \frac{\bar{C}}{\bar{R}} + \left(\frac{V}{const}\right)^{-\bar{\Gamma}_2}$,

Moreover, using boundary conditions, it is immediate to see that there exists a $\bar{V} \in (V_b, \infty)$ such that $\bar{P} > \underline{P}$ for all $V \in [\bar{V}, \infty)$. Since it does not hold that $\bar{P} > \underline{P}$ almost everywhere, there must exist a non-degenerate interval such that $\bar{P} < \underline{P}$ over that interval. Hence, by continuity, there exists a point $V^\star$ such that $\bar{P}(V^\star) = \underline{P}(V^\star)$ and moreover, by concavity of $\bar{P}$ and $\underline{P}$, it holds that $0 > \underline{P}'' > \bar{P}''$ for almost all $V > V^\star$, which is a contraction. Therefore, P^{st} is increasing in $C(1)/R(1)$.

Finally, direct calculations lead to:

$$\frac{\partial(C(1)/R(1))}{\partial\beta} > 0 \Longleftrightarrow \frac{\partial(C(1)/R(1))}{\partial\theta} > 0 \Longleftrightarrow (1-M)u + M\kappa > c/r,$$

which always hold as $c/r < \kappa < u$.

A.5 Proof of proposition 4

Suppose by contradiction that there exists a no-trade equilibrium. This implies that $\pi(V) = 0$ for almost all $V \in [V_e, \infty)$. Using the boundary conditions, it must hold that $\lim_{V\to\infty} P(V) = c/r$. Hence, by continuity of the price, there exists a non-degenerate interval $I \subseteq [V_b, \infty)$ such that $P(V) > \kappa$ for almost all $V \in I$. By lemma 1, it must be that $\pi(V) = 1$ for almost all $V \in I$, which contradicts $\pi(V) = 0$ for almost all $V \in [V_e, \infty)$.

A.6 Proof of proposition 5

Using Ito's lemma to expand the equity value:

$$
\begin{aligned}
E_t &\approx \sup_{\tau\in\mathcal{T}} \mathbb{E}_V \left[\int_0^h e^{-rt} NC(V_t)dt + e^{-rh}E(V_h)\right] \\
&\approx \sup_{\tau\in\mathcal{T}} \left\{ NC(V)h + \frac{1}{1+rh}\left[E(V) + E'(V)\mu V h + \frac{1}{2}E''(V)\sigma^2 V^2 h\right]\right\}
\end{aligned}
$$

Let $E_t = w(V_t)$. Rearrange terms, divide by h, and let $h \to 0$ to get:

$$
rw(V) = \sup_{\tau} \left\{ V - \zeta c + \frac{M}{T}\left(\tilde{P}(V) - p\right) + \mu V w'(V) + \frac{\sigma^2}{2}V^2 w''(V)\right\}
$$

Standard verification arguments leads to $\tau = \inf\{t : V_t \leq V_b\}$ and optimality implies smooth pasting. See e.g. Duffie and Lando (2001) (proposition 2.1) for details.

A.7 Proof of proposition 6 and corollary 3

By taking the change of variable $y \equiv \ln(V)$, equation (HJBE) becomes:

$$
rE(y) = e^y - \zeta c + mE'(y) + \frac{\sigma^2}{2}E''(y) + \nu[P(y) - p],
$$

where $\nu := \frac{M}{T}\left(1 - \frac{1}{T}\right)$ and $m := \mu - \frac{\sigma^2}{2}$. The homogeneous solutions of the ODE are $y_1(y) = e^{\gamma_1 y}$ and $y_2(y) = e^{-\gamma_2 y}$. Since $r > \mu$ then we have that $\gamma_1 > 1 > 0 > -\gamma_2$. For the non-homogeneous case, the multiplicative parameters are constant and P is bounded and smooth, hence the variation of parameters can be applied. First, the Wronskian W is:

$$
\begin{aligned}
W(s) &= y_1(s)y_2'(s) - y_1'(s)y_2(s) \\
&= -(\gamma_1 + \gamma_2)e^{(\gamma_1 - \gamma_2)s} \\
&= -(\gamma_1 + \gamma_2)y_1(s)y_2(s)
\end{aligned}
$$

Denote the non-homogeneous term with $g(s) := -e^s + \zeta c - \nu[P(s) - p]$. By variation of parameters, a particular solution $y_p(y)$ is:

$$\begin{aligned} y_p(y) &= \int \frac{2}{\sigma^2} g(s) \frac{y_1(s)y_2(y) - y_1(y)y_2(s)}{W(s)} ds \\ &= \frac{2}{\sigma^2} \int g(s) \frac{y_2^{-1}(s)y_2(y) - y_1(y)y_1^{-1}(s)}{-(\gamma_1 + \gamma_2)} ds \\ &= \frac{2}{\sigma^2} \int g(s) \frac{e^{\gamma_2 s} e^{-\gamma_2 y} - e^{\gamma_1 y} e^{-\gamma_1 s}}{-(\gamma_1 + \gamma_2)} ds \end{aligned}$$

Starting from the safe-trade equilibrium, let us replace $g(s)$ and solve the integral term by term. From $-e^s$ we get:

$$\begin{aligned} \int -e^s \frac{e^{\gamma_2 s} e^{-\gamma_2 y} - e^{\gamma_1 y} e^{-\gamma_1 s}}{-(\gamma_1 + \gamma_2)} ds = &\int \frac{e^{(1+\gamma_2)s} e^{-\gamma_2 y} - e^{(1-\gamma_1)s} e^{\gamma_1 y}}{\gamma_1 + \gamma_2} ds \\ = &\frac{1}{\gamma_1 + \gamma_2} \left[\left(\frac{e^y}{1 + \gamma_2} + K_2 e^{-\gamma_2 y} \right) - \left(\frac{e^y}{1 - \gamma_1} + K_1 e^{\gamma_1 y} \right) \right] \\ = &K_1 e^{\gamma_1 y} + K_2 e^{-\gamma_2 y} - \frac{e^y}{(1 - \gamma_1)(1 + \gamma_2)} \end{aligned}$$

where K_i are constants to be determined with boundary conditions. From the constant term $\zeta c + \nu p$ we get:

$$\begin{aligned} \int (\zeta c + \nu p) \frac{e^{\gamma_2 s} e^{-\gamma_2 y} - e^{\gamma_1 y} e^{-\gamma_1 s}}{-(\gamma_1 + \gamma_2)} ds = &- \frac{\zeta c + \nu p}{\gamma_1 + \gamma_2} \left[\left(\frac{1}{\gamma_2} + K_2 e^{-\gamma_2 y} \right) + \left(\frac{1}{\gamma_1} + K_1 e^{\gamma_1 y} \right) \right] \\ = &K_1 e^{\gamma_1 y} + K_2 e^{-\gamma_2 y} - \frac{\zeta c + \nu p}{\gamma_1 \gamma_2} \end{aligned}$$

From the price $-\nu P(s)$ we get for almost every $y \in (\ln V_b, \ln V_e)$:

$$\begin{aligned} \int -\nu P(s) \frac{e^{\gamma_2 s} e^{-\gamma_2 y} - e^{\gamma_1 y} e^{-\gamma_1 s}}{-(\gamma_1 + \gamma_2)} ds &= \int \nu (a_0 + a_1 e^{\gamma_1 s} + a_2 e^{-\gamma_2 s}) \frac{e^{\gamma_2 s} e^{-\gamma_2 y} - e^{\gamma_1 y} e^{-\gamma_1 s}}{\gamma_1 + \gamma_2} ds \\ &= K_1 e^{\gamma_1 y} + K_2 e^{-\gamma_2 y} + \frac{a_0 \nu}{\gamma_1 \gamma_2} - \frac{\nu a_1}{\gamma_1 + \gamma_2} e^{\gamma_1 y} y + \frac{\nu a_2}{\gamma_1 + \gamma_2} e^{-\gamma_2 y} y \end{aligned}$$

For almost every $y \in [\ln V_e, \infty)$:

$$\int -\nu P(s) \frac{e^{\gamma_2 s} e^{-\gamma_2 y} - e^{\gamma_1 y} e^{-\gamma_1 s}}{-(\gamma_1 + \gamma_2)} ds = \int \nu (A_0 + A_2 e^{-\Gamma_2 s}) \frac{e^{\gamma_2 s} e^{-\gamma_2 y} - e^{\gamma_1 y} e^{-\gamma_1 s}}{\gamma_1 + \gamma_2} ds$$
$$= C_1 e^{\gamma_1 y} + C_2 e^{-\gamma_2 y} + \frac{A_0 \nu}{\gamma_1 \gamma_2} - \frac{A_2 \nu e^{-\Gamma_2 y}}{(\Gamma_2 + \gamma_1)(\Gamma_2 - \gamma_2)}$$

Hence, collecting all the terms:

$$E^{st}(y) = \begin{cases} K_1 e^{\gamma_1 y} + K_2 e^{-\gamma_2 y} + \frac{2}{\sigma^2} \left[-\frac{\zeta c - \nu(a_0 - p)}{\gamma_1 \gamma_2} - \frac{e^y}{(1-\gamma_1)(1+\gamma_2)} - \frac{\nu a_1}{\gamma_1 + \gamma_2} e^{\gamma_1 y} y + \frac{\nu a_2}{\gamma_1 + \gamma_2} e^{-\gamma_2 y} y \right], \\ \quad \text{for } \; y \in (\ln V_b, \ln V_e) \\ \\ C_1 e^{\gamma_1 y} + C_2 e^{-\gamma_2 y} + \frac{2}{\sigma^2} \left[-\frac{\zeta c - \nu(A_0 - p)}{\gamma_1 \gamma_2} - \frac{e^y}{(1-\gamma_1)(1+\gamma_2)} - \frac{A_2 \nu e^{-\Gamma_2 y}}{(\Gamma_2 + \gamma_1)(\Gamma_2 - \gamma_2)} \right], \\ \quad \text{for } \; y \in [\ln V_e, \infty) \end{cases}$$

Now notice that $\gamma_1 \gamma_2 = \frac{2r}{\sigma^2}$, $-(1-\gamma_1)(1+\gamma_2) = \frac{2(r-\mu)}{\sigma^2}$, $-(\gamma_1 + \gamma_2) = \frac{2m}{\sigma^2}$, and $-(\Gamma_2 + \gamma_1)(\Gamma_2 - \gamma_2) = \frac{2}{\sigma^2}(r - R(1))$.

Imposing the boundary condition $\lim_{V \to \infty} \left| \frac{E(V)}{V} \right| < \infty$, taking again the change of variables, this leads to:

$$E^{st}(V) = \begin{cases} K_1 V^{\gamma_1} + K_2 V^{-\gamma_2} - \frac{\zeta c + \nu(p - a_0)}{r} + \frac{V}{r-\mu} + \frac{\nu}{m} \ln V \left(a_1 V^{\gamma_1} - a_2 V^{-\gamma_2} \right), \\ \quad \text{for } \; V \in (V_b, V_e) \\ \\ C_2 V^{-\gamma_2} - \frac{\zeta c + \nu(p - A_0)}{r} + \frac{V}{r-\mu} + \frac{A_2 \nu V^{-\Gamma_2}}{r - R(1)}, \\ \quad \text{for } \; V \in [V_e, \infty) \end{cases} \tag{1}$$

Imposing the remaining boundary conditions $E(V_b) = 0$, $\underset{V \to V_e^-}{E(V)} = \underset{V \to V_e^+}{E(V)}$, and $\underset{V \to V_e^-}{E'(V)} = \underset{V \to V_e^+}{E'(V)}$ gives the remaining constants K_1, K_2, C_2.

Let us now move to the no-trade equilibrium. Imposing the boundary condition $\lim_{V \to \infty} \left| \frac{E(V)}{V} \right| < \infty$, the equity value is given by:

$$E^{nt}(V) = C_2 V^{-\gamma_2} - \frac{\zeta c + \nu(p - \frac{c}{r})}{r} + \frac{V}{r-\mu} - \frac{\nu}{m} \ln V \left[\frac{(1-\alpha)}{r-\mu} V_b - \frac{c}{r} \right] \left(\frac{V}{V_b} \right)^{-\gamma_2} \tag{2}$$

The constant C_2 is pinned down imposing $E(V_b) = 0$:

$$C_2 = \frac{\zeta c + \nu(p - \frac{c}{r})}{r} V_b^{\gamma_2} - \frac{V_b^{1+\gamma_2}}{r-\mu} + \frac{\nu}{m}\left[\frac{(1-\alpha)}{r-\mu}V_b - \frac{c}{r}\right] V_b^{\gamma_2} \ln V_b.$$

The smooth pasting condition leads to:

$$\frac{1+\gamma_2}{r-\mu} - \frac{\zeta c + \nu(p - \frac{c}{r})}{r}\frac{\gamma_2}{V_b} + \frac{\nu \frac{c}{r}}{\mu - \sigma^2/2}\frac{1}{V_b} - \frac{(1-\alpha)}{r-\mu}\frac{\nu}{\mu - \sigma^2/2} = 0.$$

Solving for V_b leads to the equation for the threshold in proposition 6.

Let us now show that $E^{st} > E^{nt}$ and $V^{st} < V^{nt}$.[10] Fix a default boundary V_b for both equilibria. From corollary 1, $P^{st}(V;V_b) > P^{nt}(V;V_b)$ uniformly in $V > V_b$. Let $E(V,V_b)$ denote the equity valuation when the state is V and the default boundary is V_b, potentially different from the optimal one. The Feynman-Kac formula and equation (HJBE) imply that $E^{st}(V;V_b) > E^{nt}(V;V_b)$ uniformly in $V > V_b$. Now consider the two different boundaries V_b^{st}, V_b^{nt} and suppose by contradiction that $V_b^{st} \geq V_b^{nt}$. Using the boundary conditions:

$$E^{st}(V_b^{st};V_b^{st}) = E^{nt}(V_b^{nt};V_b^{nt}) = 0.$$

Optimality of the default boundary implies:

$$0 = E^{st}(V_b^{st};V_b^{st}) > E^{st}(V_b^{st};V_b^{nt}) > E^{nt}(V_b^{st};V_b^{nt})$$

which is a contradiction because it cannot be that $E^{nt}(V;V_b^{nt}) < 0$ at any $V \geq V_b^{nt}$. Therefore $V_b^{st} < V_b^{nt}$. Finally using optimality for all $V > V_b^{st}$:

$$E^{st}(V;V_b^{st}) > E^{st}(V;V_b^{nt}) > E^{nt}(V;V_b^{nt}),$$

which implies that $E^{st} > E^{nt}$ for almost all $V > V_b^{st}$.

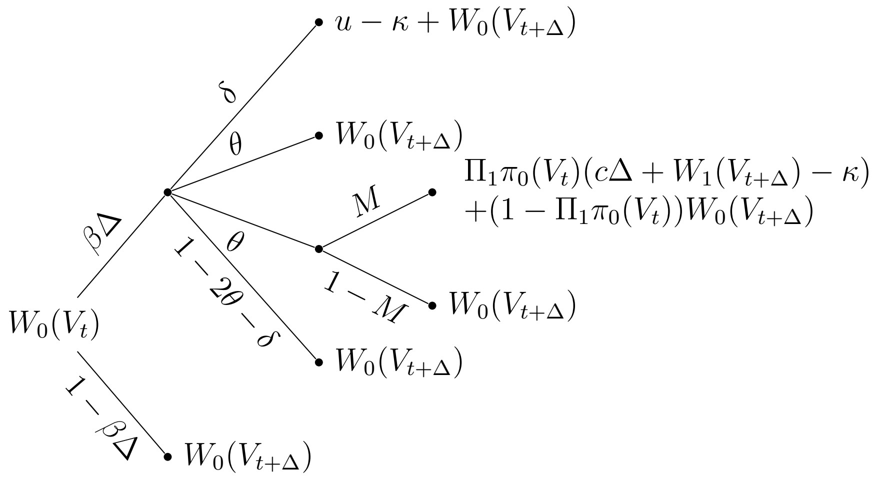
Finally, the proof of corollary 3 follows from the same argument. Consider a reduction of the search friction $\beta' > \beta$. This increases the price as in corollary 2 for a given default threshold, and the default threshold must decrease because:

$$0 = E^{st}(V_b';V_b',\beta') > E^{st}(V_b';V_b,\beta') > E^{st}(V_b';V_b,\beta)$$

where V_b' is the optimal default threshold associated to β', and V_b is the optimal default threshold associated to β.

[10] See also proposition 2 in He and Xiong (2012c) for a proof along similar lines.

B Additional Figures



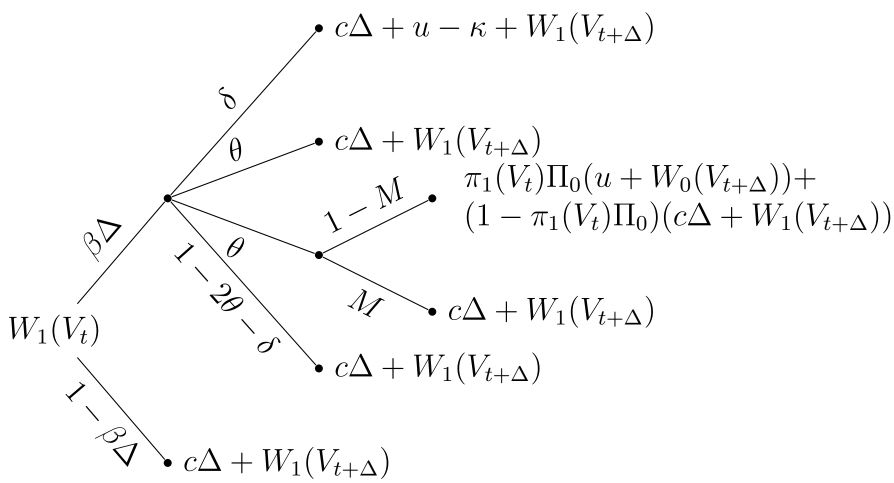


Figure 4: Value functions for the two agents.

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